

Computing (r, Q) Policy for an Inventory System with Limited Sharable Resource

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Abstract: This paper deals with an inventory system with limited resource for a single item or multiple items under continuous review (r, Q) policies. For the single-item system with a stochastic demand, a constant lead time, and limited resource, it is shown that an existing algorithm can be applied for finding an optimal (r, Q) policy that minimizes the expected system costs. For the multi-item system with limited and sharable-common resource, each item faces a stochastic demand and a constant lead time. An optimization problem is formulated for finding optimal (r, Q) policies for all items, which minimize the expected system costs. Bounds on the optimal solution (policy) and lower bounds on the minimum expected system costs are obtained. Based on the bounds, a solution approach is proposed to find an optimal or near-optimal solution and to evaluate the quality of the solution. Numerical examples demonstrate that the solution approach finds optimal or near-optimal policies for most of the tested cases.

Keywords: Inventory, (r, Q) policy, stochastic demand, sharable-common resource, algorithm.

1. Introduction

The (r, Q) policy is a typical policy in inventory management for systems with stochastic or deterministic demands. For a system with an (r, Q) policy, the inventory is reviewed continuously, and whenever the inventory position drops to or below the reorder point r , a quantity of Q units of goods is ordered with a lead time to delivery. During the lead time, shortages to customer demands can be backordered or lost. An important issue in systems with (r, Q) policies is to determine the values of r and Q for optimal inventory management. Usually, in such an inventory system, three types of costs are considered: set-up cost for ordering goods, holding cost for on-hand inventory, and penalty cost for customer backorders. The optimal values of r and Q are related to the minimization of the expected total costs. Most studies focus on models with backorder case for shortages. An efficient algorithm has been proposed to find optimal policies for discrete inventory systems in Federgrun and Zheng [4]. Other work on (r, Q) policies includes Axsäter [1], Hadley and Whitin [5], Lau, et al. [9], Sahin [12], Sivazlian [13], Zheng [15]. For models with consideration of lost sales for shortages, studies can be referred to, e.g., Hill and Johansen [7] (and references therein). For variety modeling and analysis of (r, Q) policies including optimization algorithms, Zipkin [16] has provided a comprehensive and detailed introduction (also see references therein). The above works are focused on single-item systems without resource constraints. Our study, based on (r, Q) policies with backorder case, discusses single-item and multi-item systems with resource constraints.

It is common that most actual inventory systems are designed for storing goods of different items. For instance, the distribution center of a chain-store manages its inventory system for a lot of items. Similar modes exist in wholesalers, third-party logistics centers, as well as department stores. Furthermore, for many such inventory systems, resource available for inventory management is limited. When multiple items are present and resource is limited, the utilization of resource may arise as a major issue and should be considered in inventory management.

Usually, there exist two kinds of resources: exclusive-separate resource and sharable-common resource.

The exclusive-separate resource can be the storage-space in a warehouse. This kind of resource is utilized in the following manner. Goods of different items exhibit individual characteristics in, for example, weight, shape and volume. A particular item may need its own shelves and/or special equipments for placing its goods. Thus, each item must possess its own space to store the corresponding goods. The space allocated to a particular item is then exclusively occupied and used by this item. For inventory systems with continuous review (r, Q) policies, the maximum on-hand inventory is possibly $r + Q$. Consequently, the requirement for the amount of resource is $r + Q$ for a particular item. Two major issues in such inventory systems are the allocation of the resource to, and the determination of the values of r and Q for, individual items, under the constraint that the total requirement for the resource is less than or equal to the total amount of the resource available. Zhao, et al. [14] has studied a model with a resource constraint for on-hand inventory, where an optimization problem for the optimal policy is introduced and solved.

The sharable-common resource can be investments or capital. As described by Minner and Silver [10], "Practical applications are budget constraints, where the total amount of capital tied up in inventories at any time is limited by corporate strategy in industry or by law/regulation in government/military applications". See also Betts and Johnston [2] for similar description. Thus, the consideration of the sharable-common resource in inventory management is well motivated by practical concerns. Unlike the exclusive-separate resource, the sharable-common resource can be commonly shared among all items. Such resource is used and then occupied, when an order is placed. Therefore, the resource is related to the inventory position rather than the maximum on-hand inventory. For an (r, Q) policy, the inventory position is a random variable possibly taking negative or positive values. Whenever the inventory position is positive, a corresponding amount of the resource is occupied. If the amount of the resource available can cover the summation of the maxima of inventory positions over all items, then no risk in resource shortages exists in the system. However, the utilization ratio of the resource can be low and the system may not be economically sound. For example, consider a system with 10 items. Assume that each

item is at the maximum in its inventory position with probability 0.1 in the steady state. Then, at an arbitrary time, the probability that all items coincide at the maxima in their inventory positions is 10^{-10} , which is negligible. (See also Betts and Johnston [2] for more examples.) On the other hand, if we increase the inventory in order to improve the utilization ratio of the resource, the risk in resource shortages will increase. Therefore, the inventory management should take into consideration of the trade-off between the utilization ratio of the resource and the risk in resource shortages. In general, it should not be necessary for the available resource to cover the maxima of inventory positions of all items because these maxima may seldom coincide simultaneously. Whenever a resource shortage occurs, the system can rent or loan extra resource temporarily and a shortage cost is incurred. The shortage cost may be offset by the benefit of keeping less sharable-common resource.

Inventory systems with limited and sharable-common resource exist widely in the real logistics field, but research results are expected to instruct practices. Minner and Silver [10] have considered an inventory system with continuous review dynamic replenishment policy, in which lead times are zero for replenishing goods and backorders are not permitted. Therefore, replenishments are carried out when the on-hand inventory drops to zero. With such assumptions, the on-hand inventory and the inventory position become the same. They formulate the problem as a semi-Markov decision process and propose heuristics to solve the problem by referring to the economic ordering quantity (EOQ). On the other hand, Betts and Johnston [2] have formulated an approximate objective function, rather than the exact cost function to be optimized. They develop an approximate solution approach to solve their model. Other existing work deals with inventory control problems with limited resource for newsboy models or for models with periodic review order-up-to policies, e.g., Erlebacher [3], Lau and Lau [8], and Hausman, et al. [6], which are different from models with continuous review (r, Q) policies.

In this paper, we study an inventory system with limited resource for a single item and multiple items under continuous review (r, Q) policies. For individual items, lead times are constant and backorders are allowed. We take into consideration the exact cost objective function with the resource shortage cost. In Section 2, the single-item system with limited resource is analyzed. In Section 3, we introduce the multi-item system with limited and sharable-common resource and formulate an optimization problem for optimal (r, Q) policies. Section 4 obtains bounds for the optimization problem. Based on the bounds, Section 5 develops a solution approach to find an optimal or approximately optimal solution. Section 6 provides numerical evaluations for the efficiency and effectiveness of the solution approach. The paper is concluded in Section 7.

2. Single-item system with limited resource

In this section, a single-item system with unlimited resource is reviewed first. Then, a single-item system with limited resource is analyzed. The analysis in this section lays the foundation for that of a multi-item system considered in subsequent sections.

Consider a single-item system consisting of a manufacturer, an inventory system, and customers. Goods are discrete, i.e., unit quantity. Customer demands arrive at the inventory system according to a renewal process with a mean λ per time unit. Every demand requires one unit of goods, and demands that cannot be satisfied immediately are backordered. The inventory system places orders to the manufacturer. The manufacturer replenishes goods to the inventory system with a constant lead time L after receiving an order.

As in the literature, the inventory level is defined as the amount of goods on-hand minus the number of backorders. Note that the amount of goods on-hand and the number of backorders cannot be positive simultaneously at any time. Consequently, the positive inventory level refers to the amount of goods on-hand whereas the negative inventory level relates to the number of backorders. An order in replenishment is called an outstanding order. Note that there can be more than one outstanding order at a time. The *inventory position*, denoted by I , is defined as the inventory level plus the amount of goods in outstanding orders, which is a random variable.

The inventory is reviewed continuously and controlled by an (r, Q) policy. The feasible values of r are finite integers (negative, zero or positive), whereas those of Q must be finite positive integers. The (r, Q) policy is based on the inventory position and works in the following manner: whenever the inventory position I drops to or below r , the inventory system places an order to the manufacturer for an amount of Q units of goods. Since the inventory is reviewed continuously, it is easy to see that the inventory position I takes values on $\{r + 1, r + 2, \dots, r + Q\}$. A useful existing result is that, in the steady state, the inventory position I is uniformly distributed on $\{r + 1, r + 2, \dots, r + Q\}$ (see, e.g., Federgruen and Zheng [4], Hadley and Whitin [5] and Sivazlian [13]).

Three types of costs are considered: set-up cost K per order, holding cost h per unit of goods held per time unit, and penalty cost p per demand backordered per time unit. In the steady state, given the inventory position $I = y$, the summation of the expected holding and penalty costs per time unit can be expressed as

$$g(y) = h \sum_{i=0}^y (y-i) \cdot \Pr\{D=i\} + p \sum_{i=y+1}^{\infty} (i-y) \cdot \Pr\{D=i\}, \quad (2.1)$$

where D is the demand during a lead time, which has a mean λL . It is known that the function $g(y)$ is convex with respect to y , and $-g(y)$ is a unimodal function.

With unlimited resource, the expected total costs per time unit can be expressed as

$$c(r, Q) = \frac{K\lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} g(y). \quad (2.2)$$

Let $\Omega = \{(i, j) \mid -\infty < i < \infty, 1 \leq j < \infty, i \text{ and } j \text{ are integers}\}$ represent the set of all feasible (r, Q) policies.

Problem 2.1 Find (r, Q) in Ω to minimize $c(r, Q)$.

An efficient algorithm has been proposed to solve the above optimization problem by Federgruen and Zheng (1992), which can be summarized as follows.

Algorithm 2.1

Step 1. Find y^* that minimizes $g(y)$;

Step 2. Set $q_{\min} = y^*, q_{\max} = y^*$;

Step 3. Set $r = q_{\min} - 1, Q = q_{\max} - q_{\min} + 1$;

Step 4. If $\min\{g(r), g(r+Q+1)\} \geq c(r, Q)$, then stop. Otherwise, go to Step 5;

Step 5. If $g(r) \leq g(r+Q+1)$, then $q_{\min} =: q_{\min} - 1$. Otherwise, $q_{\max} =: q_{\max} + 1$. Go to step 3.

Algorithm 2.1 only requires the condition that $-g(y)$ in equation (2.2) is a unimodal function, for obtaining an optimal solution to Problem 2.1 (Federgruen and Zheng [4]). It can be shown that general convex functions $h(\cdot)$ and $p(\cdot)$ can guarantee the unimodality of $-g(y)$.

The final resultant policy obtained by the algorithm, denoted by $(\tilde{r}, \tilde{Q}) \in \Omega$, is an optimal solution of Problem 2.1.

Now, we introduce a single-item system with limited resource for goods in on-hand inventory and outstanding orders. The system operates in the same way as the one with unlimited resource, but may have resource shortage cost. When the inventory system places an order to the manufacturer, the resource is used and then occupied. If an arriving customer is satisfied immediately by a unit of goods in on-hand inventory, the customer pays for the resource and hence the corresponding resource occupied by the goods is released. If an arriving customer cannot be satisfied immediately due to no on-hand inventory but a unit of goods in outstanding orders can be assigned to the customer, the customer also pays for the resource and hence the corresponding resource occupied by the goods is released. We assume that a unit of goods in outstanding orders can be assigned to at most one customer. The above mechanism means that the resource is occupied only by the goods on-hand and the unassigned goods in outstanding orders. Thus, the amount of the resource occupied depends on the inventory position and their relationship is explained explicitly in below.

Assume that one unit of goods occupies s units of the resource. If the inventory position I is always nonnegative (i.e., $r \geq -1$), the amount of unassigned goods is I and, consequently, the amount of the resource occupied is given by sI . On the contrary, if the inventory position I is always nonpositive (i.e., $r + Q \leq 0$), there is no unassigned goods and, therefore, no resource is occupied at any time. If the inventory position can be either negative and positive (i.e., $r < -1$ and $r + Q > 0$), the amount of unassigned goods is given by $I^+ = \max\{0, I\}$ and the amount of the resource occupied is sI^+ . Combining all the three cases, the amount of the resource occupied is given by sI^+ .

For a given (r, Q) policy, the maximum of inventory position is $(r+Q)^+$ and the resource occupied can be up to $s(r+Q)^+$. If the resource available is not enough to cover the maximum of resource occupied, resource shortage may occur. When a resource shortage occurs, the system has to rent or loan extra resource, and a resource shortage cost that is proportional to the quantity of the temporarily rented or loaned resource is incurred.

Assume that the amount of the available resource is w units, and the resource shortage cost of one unit of extra resource is a . A resource shortage occurs when $sI^+ > w$. Thus, in general, the amount of the resource shortage is given by $(sI^+ - w)^+$. In addition to the cost given in equation (2.2), the system has the expected resource shortage cost per time unit given by $aE(sI^+ - w)^+$. Since $aE(sI^+ - w)^+ = E(asI^+ - aw)^+$, without loss of generality, we rewrite as as s and aw as w . Consequently, under the (r, Q) policy, the expected system costs per time unit can be given as

$$C(r, Q) = \frac{K\lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} g(y) + E(sI^+ - w)^+, \quad (2.3)$$

Since the inventory position I is uniformly distributed on $\{r+1, r+2, \dots, r+Q\}$, we obtain

$$C(r, Q) = \frac{K\lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} G(y), \quad (2.4)$$

where $G(y) = g(y) + (sy^+ - w)^+$.

Problem 2.2 Find (r, Q) in Ω to minimize $C(r, Q)$.

Let (r^*, Q^*) denote the optimal solution of Problem 2.2.

Proposition 2.1 Algorithm 2.1 can be applied to solve Problem 2.2.

Proof. It is sufficient to show that $-G(y) = -\left[g(y) + (sy^+ - w)^+\right]$ is a unimodal function, in order to use Algorithm 2.1 to obtain an optimal solution to Problem 2.2. The desired property of $G(y)$ holds, because $(sy^+ - w)^+$ is a non-decreasing and convex function with respect to y . The proof is then completed.

3. Multi-item system with limited and sharable-common resource

Assume that there are M items. Each item possesses its own attributes, such as its customers' demand, its (r, Q) policy, and its lead time. The total amount of the resource is W units, which are commonly shared by all items. For $m = 1, \dots, M$, we assume that one unit of goods of item m requires s_m units of the resource. Among M items, demands are mutually independent. The demand D_m of item m during its lead time L_m is stochastic with mean $\lambda_m L_m$, where λ_m is the demand per time unit.

In this section, we use subscript m to refer to the index of item m . Let $\mathbf{r} = (r_1, \dots, r_M)$ and $\mathbf{Q} = (Q_1, \dots, Q_M)$. A system policy, denoted by (\mathbf{r}, \mathbf{Q}) , is formed by the set of individual policies $\{(r_m, Q_m) | 1 \leq m \leq M\}$ over all items. Let $\mathbf{I} = (I_1, \dots, I_M)$, where I_m is the inventory position of item m . Since I_m has a uniform distribution on $\{r_m + 1, r_m + 2, \dots, r_m + Q_m\}$, in the steady state, the random vector \mathbf{I} corresponds to (\mathbf{r}, \mathbf{Q}) . The inventories of all items are managed independently. We assume that whenever a resource shortage occurs, an equivalent amount of extra resource is rented or loaned. Thus, I_1, \dots, I_M are independent random variables.

It is easy to see that the maximum amount of the resource occupied by item m is $s_m(r_m + Q_m)^+$. For a given system policy (\mathbf{r}, \mathbf{Q}) , the summation of the maximum amount of the resource occupied by all items is $\sum_{m=1}^M s_m(r_m + Q_m)^+$. If $\sum_{m=1}^M s_m(r_m + Q_m)^+ \leq W$, i.e., there is sufficient resource, each item can adopt its own (r, Q) policy. For such a case, the system with limited resource is equivalent to the one with unlimited resource. If $\sum_{m=1}^M s_m(r_m + Q_m)^+ > W$, resource shortage may occur and the shortage cost may be incurred. Since the inventory position of item m is I_m , the total amount of the resource occupied by all items is $\sum_{m=1}^M s_m I_m^+$. The amount of the resource shortage is then given by $\left(\sum_{m=1}^M s_m I_m^+ - W\right)^+$. As described in section 2, the cost coefficient a of the extra resource can be incorporated into $\{s_1, s_2, \dots, s_M\}$ and W . The expected system costs per time unit can then be expressed as

$$C(\mathbf{r}, \mathbf{Q}) = \sum_{m=1}^M c_m(r_m, Q_m) + E \left(\sum_{m=1}^M s_m I_m^+ - W \right)^+, \quad (3.1)$$

where

$$c_m(r_m, Q_m) = \frac{K_m \lambda_m}{Q_m} + \frac{1}{Q_m} \sum_{y=r_m+1}^{r_m+Q_m} g_m(y), \quad (3.2)$$

and $g_m(y)$ is defined by equation (2.1) with h_m , p_m and D_m for item m .

Let Ω^M denote the Cartesian product of M copies of Ω . An optimization problem is then formulated as follows.

Problem 3.1 Find $\{(r_m, Q_m), m = 1, \dots, M\}$ in Ω^M to minimize $C(\mathbf{r}, \mathbf{Q})$.

Let $(\mathbf{r}^*, \mathbf{Q}^*)$ denote the optimal solution of Problem 3.1.

To the best of the authors' knowledge, no algorithm has been developed for the above optimization problem. Obviously, we can first find the optimal policies for individual items with unlimited resource by using Algorithm 2.1, i.e., $(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}}) = \{(\tilde{r}_m, \tilde{Q}_m) \mid 1 \leq m \leq M\}$. Then, we check whether or not the condition $\sum_{m=1}^M s_m (\tilde{r}_m + \tilde{Q}_m)^+ \leq W$ is satisfied. If it is true, we have $E \left(\sum_{m=1}^M s_m \tilde{I}_m^+ - W \right)^+ = 0$ and the optimal system policy for the system with limited resource is $(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}})$ as well. Otherwise, the optimal system policy may be different from $(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}})$. Since it is difficult to solve Problem 3.1 directly, a solution approach is developed in the subsequent sections.

4. Bounds for multi-item system

In this section, lower and upper bounds on the optimal solution of Problem 3.1 and lower bounds on the minimum expected system costs of Problem 3.1 are obtained through the following analysis.

Definition 4.1 For vectors $\mathbf{r}^1 = (r_1^1, \dots, r_M^1)$, $\mathbf{Q}^1 = (Q_1^1, \dots, Q_M^1)$, $\mathbf{r}^2 = (r_1^2, \dots, r_M^2)$ and $\mathbf{Q}^2 = (Q_1^2, \dots, Q_M^2)$, $(\mathbf{r}^1, \mathbf{Q}^1)$ is larger than or equal to $(\mathbf{r}^2, \mathbf{Q}^2)$, written as $(\mathbf{r}^1, \mathbf{Q}^1) \geq (\mathbf{r}^2, \mathbf{Q}^2)$, if $r_m^1 \geq r_m^2$ and $r_m^1 + Q_m^1 \geq r_m^2 + Q_m^2$ for all $m = 1, \dots, M$.

For item m , define the following function for a given $(\mathbf{r}^0, \mathbf{Q}^0)$ with corresponding \mathbf{I}^0 ,

$$\begin{aligned} f_m \left[(r_m, Q_m) \middle| (\mathbf{r}^0, \mathbf{Q}^0) \right] &= c_m(r_m, Q_m) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^0)^+ + s_m I_m^+ - W \right]^+ \\ &= \frac{\lambda_m K_m}{Q_m} + \frac{1}{Q_m} \sum_{y=r_m+1}^{r_m+Q_m} \left[g_m(y) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^0)^+ + s_m y^+ - W \right]^+ \right]. \end{aligned} \quad (4.1)$$

It can be easily verified that $g_m(y) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^0)^+ + s_m y^+ - W \right]^+$ is convex with

respect to y , and $-\left[g_m(y) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^0)^+ + s_m y^+ - W \right]^+ \right]$ is a unimodal function.

Therefore, Algorithm 2.1 can be used for computing an optimal solution that minimizes (4.1) for given $(\mathbf{r}^0, \mathbf{Q}^0)$.

Considering all items, we define a function for given $(\mathbf{r}^0, \mathbf{Q}^0)$ corresponding to \mathbf{I}^0 as

$$F \left[(\mathbf{r}, \mathbf{Q}) \middle| (\mathbf{r}^0, \mathbf{Q}^0) \right] = \sum_{m=1}^M f_m \left[(r_m, Q_m) \middle| (\mathbf{r}^0, \mathbf{Q}^0) \right] + (1-M) E \left[\sum_{k=1}^M s_k (I_k^0)^+ - W \right]^+. \quad (4.2)$$

We introduce the following optimization problem.

Problem 4.1 For given $(\mathbf{r}^0, \mathbf{Q}^0)$ corresponding to \mathbf{I}^0 , find $\{(r_m, Q_m), m = 1, \dots, M\}$ in Ω^M to minimize $F \left[(\mathbf{r}, \mathbf{Q}) \middle| (\mathbf{r}^0, \mathbf{Q}^0) \right]$.

Since the term $(1-M) E \left[\sum_{k=1}^M s_k (I_k^0)^+ - W \right]^+$ is constant for given $(\mathbf{r}^0, \mathbf{Q}^0)$ corresponding

to \mathbf{I}^0 , it is clear that the optimal solution of Problem 4.1 can be obtained by using Algorithm 2.1 – which is equivalent to solving M single-item problems.

For Problem 4.1, we call $(\hat{\mathbf{r}}, \hat{\mathbf{Q}})$ a *stationary solution* if

$\min_{(\mathbf{r}, \mathbf{Q}) \in \Omega^M} F \left[(\mathbf{r}, \mathbf{Q}) \middle| (\hat{\mathbf{r}}, \hat{\mathbf{Q}}) \right] = F \left[(\hat{\mathbf{r}}, \hat{\mathbf{Q}}) \middle| (\hat{\mathbf{r}}, \hat{\mathbf{Q}}) \right]$, i.e., given $(\hat{\mathbf{r}}, \hat{\mathbf{Q}})$, the resultant optimal solution of

the problem is (\hat{r}, \hat{Q}) itself.

Proposition 4.1 For given (r^ζ, Q^ζ) and (r^β, Q^β) , let (r^a, Q^a) and (r^b, Q^b) be the solutions of $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (r^\zeta, Q^\zeta)]$ and $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (r^\beta, Q^\beta)]$, respectively. If $(r^\zeta, Q^\zeta) \geq (r^\beta, Q^\beta)$, then $(r^a, Q^a) \leq (r^b, Q^b)$.

Proof. See Appendix.

Denote by $(-\infty, I)$ the policy with $r_m = -\infty$ and $Q_m = 1$ for $m = 1, \dots, M$, which can be regarded as the smallest solution on Ω^M . It is not difficult to show that the solution of $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (-\infty, I)]$ is (\tilde{r}, \tilde{Q}) .

Now, we are ready to describe an iterative procedure that generates two sequences of solutions to Problem 4.1. Suppose that (\hat{r}, \hat{Q}) is a stationary solution. Taking $(-\infty, I)$ as (r^0, Q^0) in Problem 4.1, a new solution (\ddot{r}, \ddot{Q}) is obtained. Since $(-\infty, I) \leq (\hat{r}, \hat{Q})$, by Proposition 4.1, we have $(\ddot{r}, \ddot{Q}) \geq (\hat{r}, \hat{Q})$. Taking (\ddot{r}, \ddot{Q}) as (r^0, Q^0) in Problem 4.1, we obtain a new solution (r^1, Q^1) that satisfies $(-\infty, I) < (r^1, Q^1) \leq (\hat{r}, \hat{Q})$ by Proposition 4.1. Taking (r^1, Q^1) as (r^0, Q^0) in Problem 4.1, a new solution (r^2, Q^2) is generated. Since $(-\infty, I) < (r^1, Q^1)$, it holds that $(\hat{r}, \hat{Q}) \leq (r^2, Q^2) \leq (\ddot{r}, \ddot{Q})$. Repeating this process to obtain a sequence of solutions $\{(r^1, Q^1), (r^2, Q^2), \dots, (r^k, Q^k), \dots\}$. For that sequence, we have

$$(-\infty, I) < (r^1, Q^1) \leq (r^3, Q^3) \leq \dots \leq (r^{2k-1}, Q^{2k-1}) \leq \dots \leq (\hat{r}, \hat{Q}), \quad (4.3)$$

and

$$(\hat{r}, \hat{Q}) \leq \dots \leq (r^{2k}, Q^{2k}) \leq \dots \leq (r^2, Q^2) \leq (\ddot{r}, \ddot{Q}). \quad (4.4)$$

Thus, the sequence $\{(-\infty, I), (r^1, Q^1), (r^3, Q^3), \dots, (r^{2k-1}, Q^{2k-1}), \dots\}$ is a non-decreasing sequence and will be settled at a solution, denoted as $(\underline{r}, \underline{Q})$, after a finite number of iterations.

The sequence $\{(\ddot{r}, \ddot{Q}), (r^2, Q^2), (r^4, Q^4), \dots, (r^{2k}, Q^{2k}), \dots\}$, on the other hand, is a

non-increasing sequence and will be settled at a solution, denoted as (\bar{r}, \bar{Q}) , after a finite number of iterations. If $(\underline{r}, \underline{Q})$ is used as (r^0, Q^0) in problem 4.1, the resultant solution is (\bar{r}, \bar{Q}) , and vice versa. This indicates that $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (\bar{r}, \bar{Q})] = F[(\underline{r}, \underline{Q}) | (\bar{r}, \bar{Q})]$ and $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (\underline{r}, \underline{Q})] = F[(\bar{r}, \bar{Q}) | (\underline{r}, \underline{Q})]$.

It is obvious that a stationary solution (\hat{r}, \hat{Q}) satisfies $(-\infty, I) < (\hat{r}, \hat{Q}) \leq (\ddot{r}, \ddot{Q})$. The above analysis leads to the following result.

Lemma 4.1 For any stationary solution (\hat{r}, \hat{Q}) , it holds that $(\underline{r}, \underline{Q}) \leq (\hat{r}, \hat{Q}) \leq (\bar{r}, \bar{Q})$.

It is possible that Problem 4.1 may possess multiple stationary solutions. Proposition 4.1 and Lemma 4.1 imply that $(\underline{r}, \underline{Q})$ and (\bar{r}, \bar{Q}) are lower and upper bounds on all the stationary solutions.

By a comparison of equations (3.1) and (4.2), we obtain

$$C(r, Q) = F[(r, Q) | (r, Q)] , \quad (4.5)$$

from which it can be shown that the optimal solution of Problem 3.1, (r^*, Q^*) , is one of the stationary solutions of Problem 4.1. Therefore, if $(\underline{r}, \underline{Q}) = (\bar{r}, \bar{Q})$, then we have $(r^*, Q^*) = (\hat{r}, \hat{Q}) = (\underline{r}, \underline{Q})$, i.e., Problem 3.1 possesses a unique optimal solution and Problem 4.1 possesses a unique stationary solution.

Lemma 4.2 For given (r^0, Q^0) , $C(r, Q) \geq F[(r, Q) | (r^0, Q^0)]$ holds for all $(r, Q) \leq (r^0, Q^0)$.

Proof. See Appendix.

The following lower bound can be obtained.

Proposition 4.2 A lower bound of Problem 3.1 is given by

$$C(r^*, Q^*) \geq F[(\underline{r}, \underline{Q}) | (\bar{r}, \bar{Q})] . \quad (4.6)$$

Proof. Since (r^*, Q^*) is one of the stationary solutions of Problem 4.1, by Lemma 4.1, we have

$(\mathbf{r}^*, \mathbf{Q}^*) \leq (\bar{\mathbf{r}}, \bar{\mathbf{Q}})$. Then, Lemma 4.2 leads to

$$C(\mathbf{r}^*, \mathbf{Q}^*) \geq F[(\mathbf{r}^*, \mathbf{Q}^*) | (\bar{\mathbf{r}}, \bar{\mathbf{Q}})] \geq \min_{(\mathbf{r}, \mathbf{Q}) \in \Omega^M} F[(\mathbf{r}, \mathbf{Q}) | (\bar{\mathbf{r}}, \bar{\mathbf{Q}})] = F[(\underline{\mathbf{r}}, \underline{\mathbf{Q}}) | (\bar{\mathbf{r}}, \bar{\mathbf{Q}})],$$

which completes the proof.

In addition to the above lower bound, another lower bound can be obtained as follows. Recall that $(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}})$ is the optimal system policy with unlimited resource. Consider the objective function (3.1). It is straightforward to show that

$$C(\mathbf{r}^*, \mathbf{Q}^*) = \sum_{m=1}^M c_m(r_m^*, Q_m^*) + E \left(\sum_{k=1}^M s_k(I_k^*)^+ - W \right)^+ \geq \sum_{m=1}^M c_m(\tilde{r}_m, \tilde{Q}_m) + E \left(\sum_{k=1}^M s_k(\underline{I}_k)^+ - W \right)^+ \quad (4.7)$$

where I_k^* corresponds to (r_k^*, Q_k^*) and \underline{I}_k corresponds to $(\underline{r}_k, \underline{Q}_k)$.

We shall use (4.6) and (4.7) to evaluate the quality of system policies.

5. A solution approach

The following solution approach can be used to find an optimal or approximately optimal solution to Problem 3.1. The solution approach consists of two parts. First, use the iterative procedure described in Section 4 to produce $(\underline{\mathbf{r}}, \underline{\mathbf{Q}})$ and $(\bar{\mathbf{r}}, \bar{\mathbf{Q}})$ as well as to obtain the lower bound $F[(\underline{\mathbf{r}}, \underline{\mathbf{Q}}) | (\bar{\mathbf{r}}, \bar{\mathbf{Q}})]$. If $(\underline{\mathbf{r}}, \underline{\mathbf{Q}}) = (\bar{\mathbf{r}}, \bar{\mathbf{Q}})$, the unique optimal solution of Problem 3.1 is obtained as $(\mathbf{r}^*, \mathbf{Q}^*) = (\underline{\mathbf{r}}, \underline{\mathbf{Q}}) = (\bar{\mathbf{r}}, \bar{\mathbf{Q}})$. Otherwise, the solution approach goes into the second part. This part begins with the better solution among $(\underline{\mathbf{r}}, \underline{\mathbf{Q}})$ and $(\bar{\mathbf{r}}, \bar{\mathbf{Q}})$, which is called the current solution. The current solution is improved in accordance with a local search in its neighboring solutions. A neighboring solution of the current solution (\mathbf{r}, \mathbf{Q}) is defined as a solution the same as (\mathbf{r}, \mathbf{Q}) except that the policy of the item m becomes one of $(r_m + 1, Q_m)$, $(r_m - 1, Q_m)$, $(r_m, Q_m + 1)$, $(r_m, Q_m - 1)$, $(r_m + 1, Q_m + 1)$, $(r_m + 1, Q_m - 1)$, $(r_m - 1, Q_m + 1)$, and $(r_m - 1, Q_m - 1)$. Therefore, the total number of neighboring solutions of the current solution is $8 \times M$. Every time, the current solution moves to a neighboring solution with the most reduction in the cost function (3.1) among the neighboring solutions. The procedure is repeated until the current solution cannot be improved further.

The above solution approach is summarized into the following algorithm.

Algorithm 5.1

Step 1. Find (\tilde{r}, \tilde{Q}) , the optimal system policy with unlimited resource, by Algorithm 2.1;

Step 2. Let $(r^0, Q^0) = (\tilde{r}, \tilde{Q})$, $(\underline{r}, \underline{Q}) = (\tilde{r}, \tilde{Q})$, and $(\bar{r}, \bar{Q}) = (\tilde{r}, \tilde{Q})$;

Step 3. Using Algorithm 2.1, solve $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (r^0, Q^0)]$ to generate (r^1, Q^1) ;

Step 4. Using Algorithm 2.1, solve $\min_{(r, Q) \in \Omega^M} F[(r, Q) | (r^1, Q^1)]$ to generate (r^2, Q^2) ;

Step 5. If $(r^1, Q^1) = (\underline{r}, \underline{Q})$ and $(r^2, Q^2) = (\bar{r}, \bar{Q})$, then go to Step 7.

Step 6. Let $(\underline{r}, \underline{Q}) = (r^1, Q^1)$, $(\bar{r}, \bar{Q}) = (r^2, Q^2)$ and $(r^0, Q^0) = (r^2, Q^2)$. Go to Step 3.

Step 7. If $(\underline{r}, \underline{Q}) = (\bar{r}, \bar{Q})$, then stop with the optimal solution $(r^*, Q^*) = (\underline{r}, \underline{Q})$.

Step 8. Let (r, Q) be the better one among $(\underline{r}, \underline{Q})$ and (\bar{r}, \bar{Q}) , with cost $C(r, Q)$.

Step 9. Calculate $C(r', Q')$'s for all the neighboring solutions (r', Q') of (r, Q) , and denote the one with the minimum costs by $C(\hat{r}, \hat{Q})$.

Step 10. If $C(\hat{r}, \hat{Q}) \geq C(r, Q)$, then set $(\hat{r}, \hat{Q}) = (r, Q)$ as the final solution and stop. Otherwise, let $(r, Q) = (\hat{r}, \hat{Q})$, $C(r, Q) = C(\hat{r}, \hat{Q})$ and go to Step 9.

In the algorithm, steps 1 to 7 are for the iterative procedure for computing $(\underline{r}, \underline{Q})$ and (\bar{r}, \bar{Q}) , and steps 8 to 10 are for the local search for (\hat{r}, \hat{Q}) .

The final resultant solution is an optimal solution if $(\underline{r}, \underline{Q}) = (\bar{r}, \bar{Q})$; otherwise it is an approximately optimal solution (\hat{r}, \hat{Q}) whose quality (i.e., how close it is to the optimal policy) can be evaluated as follows.

For Problem 3.1, define a relative error of the expected system costs at a policy (r, Q) to that at the optimal solution (r^*, Q^*) as

$$\Delta = \frac{C(r, Q) - C(r^*, Q^*)}{C(r^*, Q^*)} \times 100\% . \quad (5.1)$$

Using lower bounds (4.6) and (4.7) to replace $C(r^*, Q^*)$ in (5.1) by

$$\max \left\{ F \left[(\underline{r}, \underline{Q}) | (\bar{r}, \bar{Q}) \right], \sum_{m=1}^M c_m (\tilde{r}_m, \tilde{Q}_m) + E \left(\sum_{k=1}^M s_k (I_k)^+ - W \right)^+ \right\}, \text{ we obtain}$$

$$\Theta = \frac{C(\mathbf{r}, \mathbf{Q}) - \max \left\{ F \left[(\underline{r}, \underline{Q}) | (\bar{r}, \bar{Q}) \right], \sum_{m=1}^M c_m (\tilde{r}_m, \tilde{Q}_m) + E \left(\sum_{k=1}^M s_k (I_k)^+ - W \right)^+ \right\}}{\max \left\{ F \left[(\underline{r}, \underline{Q}) | (\bar{r}, \bar{Q}) \right], \sum_{m=1}^M c_m (\tilde{r}_m, \tilde{Q}_m) + E \left(\sum_{k=1}^M s_k (I_k)^+ - W \right)^+ \right\}} \times 100\%. \quad (5.2)$$

Obviously, we have $\Delta \leq \Theta$, i.e., Θ is an upper bound of Δ . Since Θ can be easily calculated, it is used to evaluate the quality of the policy (\mathbf{r}, \mathbf{Q}) . We call Θ the quality index. Apparently, the smaller Θ is, the better the policy (\mathbf{r}, \mathbf{Q}) .

6. Numerical examples

A number of examples are produced to evaluate the efficiency and effectiveness of the solution approach. Parameters of these examples are randomly generated in the ranges given in Table 1.

Table 1 Parameter combinations

The number of items M	3 ~ 20
The holding cost h_m	0.1 ~ 3.0
The penalty cost p_m	$5h_m \sim 15h_m$
The set-up cost K_m	$10h_m \sim 30h_m$
The lead time L_m	1
The demand D_m	Poisson distribution with $\lambda_m = 1 \sim 13$
The resource for one unit of goods s_m	1 ~ 5
The total amount of the resource W	$0 \sim \sum_{m=1}^M s_m (\tilde{r}_m + \tilde{Q}_m)^+$

In general, it is commonly considered in logistics area that penalty cost is much high than holding cost. Thus, we reasonably set $p_m = 5h_m \sim 15h_m$. For set-up cost, no special restriction needs to be considered; it can be very large or very small even zero. However, to obtain relatively normal solutions, we set $K_m = 10h_m \sim 30h_m$ to avoid extremely large or extremely small values to other parameters. It is noted that the lead times for all cases are one, i.e., $L_m = 1$. Since the lead time affects the calculations through $\lambda_m L_m$, it is enough to choose λ_m randomly and fix L_m so as to generate demand D_m randomly. Other parameters are set without consideration of special restriction.

We produce 100 examples from the above parameter ranges for each $M = 3, \dots, 20$. Thus, there are in total $100 \times 18 = 1800$ examples. We code the algorithm using the C language programming. All calculations are carried out on a PC with 2G memory and 3.0G Pentium 4 CPU.

Listed in Table 2 are parameters of an example with $M = 10$ and a small amount of resource. Using Algorithm 5.1, we find that $(\underline{r}, \underline{Q}) = (\bar{r}, \bar{Q})$. Thus, the optimal solution is obtained. The computational results are shown in Table 3, in which the optimal solution for the case with unlimited resource (\tilde{r}, \tilde{Q}) is also presented. Note that the amount of the available resource $W = 92$ is significantly smaller than the resource required for implementing the policy (\tilde{r}, \tilde{Q}) , which is $\sum_{m=1}^{10} s_m (\tilde{r}_m + \tilde{Q}_m)^+ = 813$. The difference of $C(\tilde{r}, \tilde{Q}) = 664.87$ and $C(r^*, Q^*) = 513.60$ is relatively large. The CPU time required to obtain the solution is relatively short for this example. (All CPU times do not include the part for finding the optimal system policies with unlimited resource (\tilde{r}, \tilde{Q}) by Algorithm 2.1, because that part is standard in the literature.)

Table 2 An example with $M = 10$ and a small amount of resource

Parameters					
Item number m	h_m	p_m	K_m	λ_m	s_m
1	1.562	15.477	18.538	1.445	4
2	1.052	15.187	28.770	2.691	1
3	1.909	11.783	31.492	9.445	5
4	2.350	21.185	26.452	10.737	3
5	0.738	6.503	15.829	7.502	1
6	1.103	7.857	31.151	8.479	5
7	1.045	15.574	21.541	11.125	2
8	0.969	10.380	18.939	10.684	5
9	0.158	1.926	2.360	1.825	5
10	2.535	29.452	56.472	9.853	3
$W = 92$					

Table 3 Computational results of the example with $M = 10$ and a small amount of resource

Item number m	(\tilde{r}, \tilde{Q})	$(r^*, Q^*) = (\underline{r}, \underline{Q}) = (\bar{r}, \bar{Q})$
1	(1, 7)	(-1, 5)
2	(2, 13)	(1, 10)
3	(6, 21)	(2, 14)

4	(9 , 18)	(7 , 14)
5	(5 , 21)	(4 , 14)
6	(5 , 25)	(-2 , 16)
7	(10 , 24)	(8 , 16)
8	(9 , 23)	(3 , 14)
9	(1 , 9)	(-2 , 2)
10	(8 , 24)	(7 , 17)
	$\sum_{m=1}^{10} s_m (\tilde{r}_m + \tilde{Q}_m)^+ = 813$ $\sum_{m=1}^{10} c_m (\tilde{r}_m, \tilde{Q}_m) = 239.87$	$\Theta = 0\%$
Cost	$C(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}}) = 664.87$	$C(\mathbf{r}^*, \mathbf{Q}^*) = 513.60$
CPU time (s)		2.99

In Tables 4, parameters of an example with a large amount of resource are presented. By Algorithm 5.1, an approximately optimal solution is obtained and is shown in Table 5. The available resource for this example is $W = 454$, which is relatively close to the resource required for implementing the optimal policy for the case with unlimited resource $(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}})$, which is $\sum_{m=1}^{10} s_m (\tilde{r}_m + \tilde{Q}_m)^+ = 665$. The quality index $\Theta = 0.04\%$ implies that the policy $(\hat{\mathbf{r}}, \hat{\mathbf{Q}})$ is close to the optimal system policy $(\mathbf{r}^*, \mathbf{Q}^*)$.

Table 4 An example with $M = 10$ and a large amount of resource

Item number m	h_m	p_m	K_m	λ_m	s_m
1	2.908	23.455	55.919	5.694	1
2	2.038	11.991	51.056	12.253	2
3	0.745	10.755	11.513	12.146	5
4	1.660	10.420	42.904	12.559	3
5	2.051	19.548	29.376	10.619	2
6	0.688	5.803	12.496	3.885	4
7	0.497	3.955	5.846	12.283	1
8	1.684	22.613	34.986	9.967	2
9	0.237	2.901	4.939	1.653	5
10	2.323	31.228	23.673	4.240	2
$W = 454$					

Table 5 Computational results of the example with $M = 10$ and a large amount of resource

Item number m	$(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}})$	$(\hat{\mathbf{r}}, \hat{\mathbf{Q}})$
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1	(4 , 17)	(4 , 16)
2	(8 , 28)	(8 , 27)
3	(12 , 21)	(11 , 17)
4	(8 , 29)	(8 , 27)
5	(9 , 20)	(9 , 19)
6	(2 , 14)	(2 , 11)
7	(10 , 20)	(10 , 19)
8	(9 , 23)	(9 , 21)
9	(1 , 9)	(0 , 6)
10	(4 , 11)	(4 , 10)
	$\sum_{m=1}^{10} s_m (\tilde{r}_m + \tilde{Q}_m)^+ = 665$ $\sum_{m=1}^{10} c_m (\tilde{r}_m, \tilde{Q}_m) = 273.53$	$\Theta = 0.04\%$
Cost	$C(\tilde{\mathbf{r}}, \tilde{\mathbf{Q}}) = 284.89$	$C(\hat{\mathbf{r}}, \hat{\mathbf{Q}}) = 276.58$
CPU time (s)		40.12

Table 6 presents parameters of an example with a moderate amount of resource. Algorithm 5.1 obtains an approximately optimal solution given in Table 7. The quality index $\Theta = 6.71\%$ implies that the relative error of the objective function (3.1) at the approximately optimal solution $(\hat{\mathbf{r}}, \hat{\mathbf{Q}})$ and at the optimal solution $(\mathbf{r}^*, \mathbf{Q}^*)$ is less than or equal to 6.71%.

Table 6 An example with $M = 10$ and a moderate amount of resource

Item number m	h_m	p_m	K_m	λ_m	s_m
1	0.984	8.855	27.996	11.217	4
2	0.601	8.122	14.838	2.051	3
3	1.475	10.795	39.766	11.330	4
4	2.409	26.682	47.843	12.681	4
5	2.865	39.427	43.705	10.191	1
6	1.153	7.846	28.980	11.213	2
7	1.069	9.600	19.450	3.818	5
8	1.358	14.298	23.006	6.700	1
9	1.675	16.206	46.612	4.994	2
10	2.143	19.663	28.431	9.735	5
$W = 473$					

Table 7 Computational results of the example with $M = 10$ and a moderate amount of resource

Item number m	(\tilde{r}, \tilde{Q})	(\hat{r}, \hat{Q})
1	(8 , 29)	(7 , 19)
2	(1 , 11)	(0 , 8)
3	(8 , 28)	(7 , 21)
4	(11 , 25)	(10 , 21)
5	(10 , 19)	(9 , 20)
6	(8 , 26)	(7 , 22)
7	(2 , 14)	(1 , 10)
8	(5 , 17)	(5 , 16)
9	(3 , 19)	(3 , 16)
10	(8 , 19)	(7 , 15)
	$\sum_{m=1}^{10} s_m (\tilde{r}_m + \tilde{Q}_m)^+ = 850$ $\sum_{m=1}^{10} c_m (\tilde{r}_m, \tilde{Q}_m) = 311.89$	$\Theta = 6.71\%$
Cost	$C(\tilde{r}, \tilde{Q}) = 384.34$	$C(\hat{r}, \hat{Q}) = 333.02$
CPU time (s)		20.47

The above three examples indicate that the quality of the approximately optimal solutions has much to do with the amount of the available resource W . Generally speaking, if W is small or large compared to the amount of the resource required for implementing the optimal policy with unlimited resource, the quality of the approximately optimal solution is satisfactory (i.e., Θ is close to 0). If W is moderate, the quality of the approximately optimal solution is acceptable for many cases. To explain the observation clearly, we define the following ratio

$$\omega = \frac{W}{\sum_{m=1}^M s_m (\tilde{r}_m + \tilde{Q}_m)^+} \quad (6.1)$$

for the relationship between the amount of the resource available and the amount of resource required for implementing the optimal policy with unlimited resource (\tilde{r}, \tilde{Q}) . We plot the quality index Θ as a function of ω for $M = 5, 10, 15$ and 20 , in Figure 1. The results indicate that satisfactory solutions are obtained for ω relatively close to zero or one. Most of the solutions are also acceptable if ω is around 0.5.

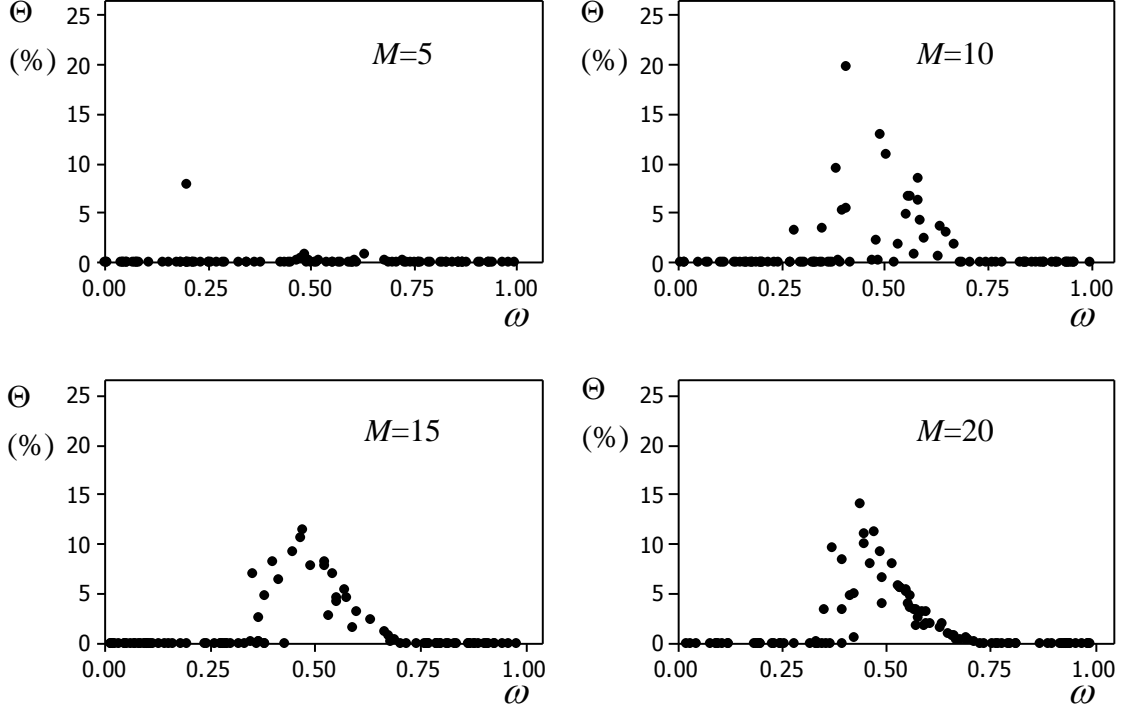


Figure 1 The relationship between Θ and ω for $M = 5, 10, 15$ and 20

For the reason about the solution quality with respect to the ratio ω , an intuitive explanation may come as follows. If the ratio is high meaning $\omega \rightarrow 1$, the optimization problem closes to the one with unlimited resource. In such case, the resource shortage $\left(\sum_{m=1}^M s_m I_m^+ - W\right)^+$ may work weakly very much, and the optimal solution may not deviate so far from the optimal solution with unlimited resource. Our solution approach performs well, because it continues search from the optimal solution with unlimited resource. On the other hand, If the ratio is low meaning $\omega \rightarrow 0$, the resource shortage $\left(\sum_{m=1}^M s_m I_m^+ - W\right)^+$ works strongly. In such case, $\left(\sum_{m=1}^M s_m I_m^+ - W\right)^+$ may play a role close to $\sum_{m=1}^M s_m I_m^+ - W$. It is easy to see our solution approach can obtain an optimal solution for the model with resource shortage $\sum_{m=1}^M s_m I_m^+ - W$. For a moderate ω , it becomes relatively difficult for our solution approach to find an optimal solution.

Among the 1800 examples, we obtain the optimal solutions for 1200 examples, satisfactory solutions (the quality index is below 5%) for 423 examples, good solutions (the quality index is between 5% and 10%) for 112 examples, and fair solutions (the quality index is between 10% and 22.6%) for 65 examples.

Figure 2 shows the quality index Θ as a function of ω for all the 1800 examples. For 552 examples with ω in the interval $[0, 0.3]$, 532 examples obtain the optimal solutions, 19

example have Θ below 5%, and one has Θ between 5% and 10%. For 658 examples with ω in the interval $[0.65, 1]$, all examples have Θ below 5%, in which 504 examples obtain optimal solutions. For 590 examples with ω in the interval $(0.3, 0.65)$, 164 examples obtain optimal solutions, 250 examples have Θ below 5%, 111 examples have Θ between 5% and 10%, and 65 examples have Θ between 10% and 22.6%.

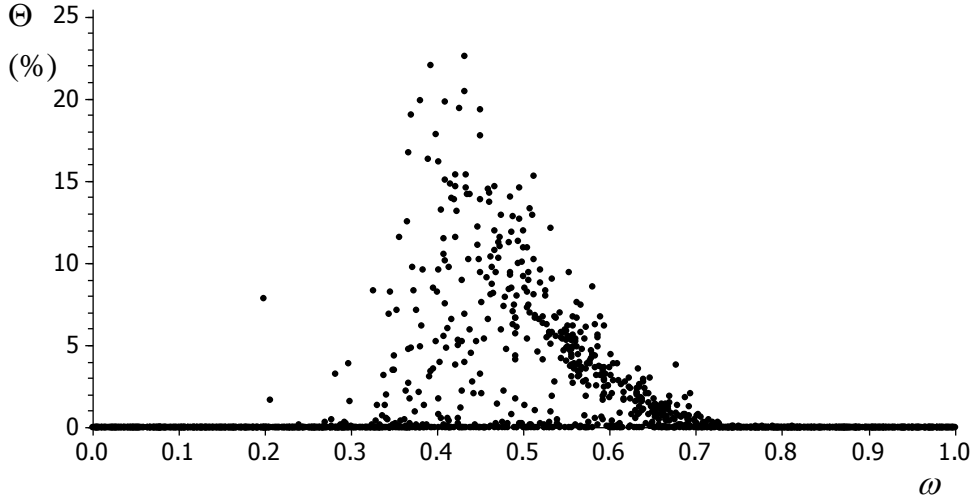


Figure 2 The relationship between Θ and ω for the 1800 examples

A little more implication may come by referring to Figure 2. We define relative change of expected system costs at the policy (r, Q) our solution approach produced and the optimal policy (\tilde{r}, \tilde{Q}) with unlimited resource as follows

$$\Gamma = \frac{C(r, Q) - C(\tilde{r}, \tilde{Q})}{C(\tilde{r}, \tilde{Q})} \times 100\% . \quad (6.2)$$

Then, it is easy to see that $\Gamma \geq \Theta$, i.e., the values of Γ are larger than those in Figure 2. The magnitude of the relative change should correlated to the ratio of the amount of the resource available to the amount of resource required for implementing the optimal policy with unlimited resource ω . In some extent, the relative change Γ with respect to ω may be referred to the figure to induce the magnitude. It is observed that considerable changes of the expected system cost exist in moderate ω . For smaller ω , such changes should be larger than those in moderate ω , although the values of Θ are close to zero.

The CPU times for all the examples are exhibited in Figure 3, in which the shortest, the longest and the average values of CPU times in second are plotted for each $M = 3, \dots, 20$. Algorithm 5.1 consists of three sub-procedures: finding optimal solution with unlimited resource; iteration to

produce $(\underline{r}, \underline{Q})$ and (\bar{r}, \bar{Q}) ; local search for improvement of the solution. Here, we do not report the CPU times of the first sub-procedure in Figure 3, because they are too short. The white histograms are CPU times of the second sub-procedure with circle marks being averages, and the shade histograms are those of the third sub-procedure with diamond marks being averages. As indicated by the Figure, the average CPU times increase approximately linearly with respect to M . From the records, we found that the actual computation time is less than two minutes for $M = 20$.

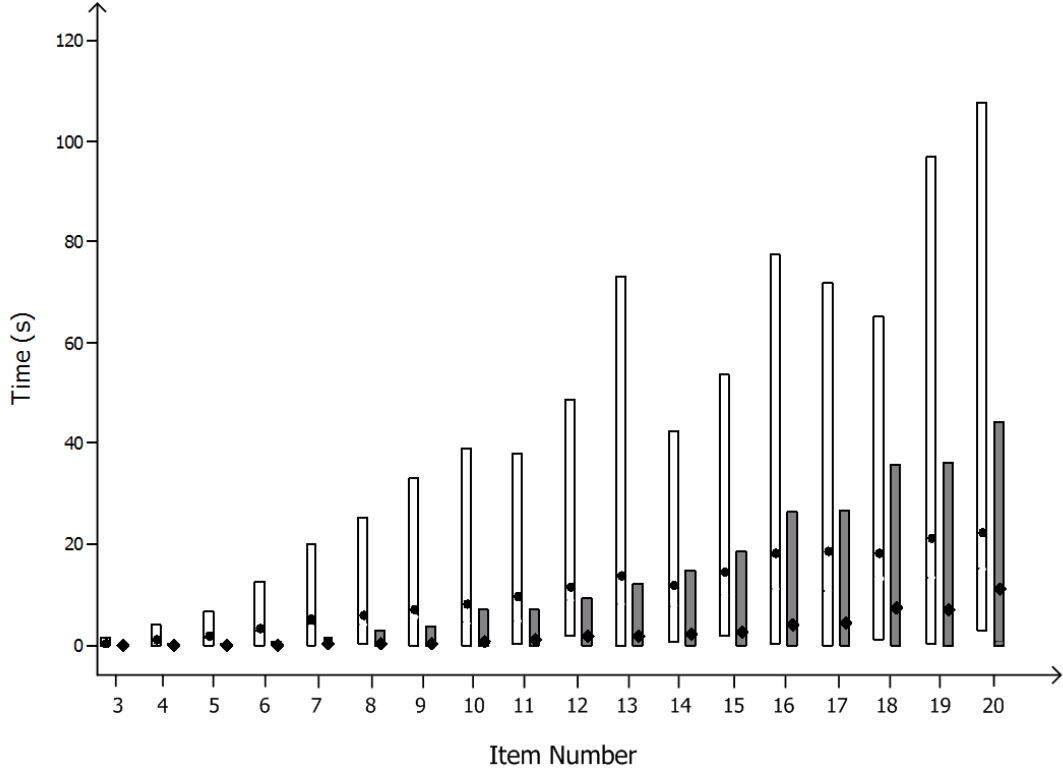


Figure 3 The CPU times for $M = 3, \dots, 20$

7. Concluding remarks

In this paper, we have shown that, for a single-item system with limited resource, the optimization problem can be solved by using an existing algorithm. For a multi-item system with limited and sharable-common resource, a solution approach is proposed to obtain an optimal or an approximately optimal solution. Numerical evaluations based on the lower bounds show that the solution approach performs well. If the ratio of the amount of the resource available to the amount of the resource required by the optimal solution with unlimited resource is relatively close to 0 or 1, the solution approach generates satisfactory solutions. For systems with a moderate amount of resource, it can also obtain good or acceptable solutions.

In our solution approach, the local search is implemented with $8M$ neighboring solutions of the current solution. In fact, for the optimization problem, the total number of neighbors is $9^M - 1$.

Possibly, the quality of solutions may be improved if we take more neighbors in the local search. We developed the local search with consideration of changing policies among two items, so the total number of neighbors is $9 \times 9M - 1 = 80M$. However, we found that the effective and the efficiency were limited. The improvement space of solutions is less than 2%, but the CUP times were several times longer. Consequently, for the part our solution approach failed to produce satisfactory solutions, we need more effective and efficient algorithms to solve the optimization problem, which can be a future work.

In the current system, an arrival customer pays for resource when the customer is satisfied immediately by a unit of goods in on-hand inventory, or when a unit of goods in outstanding orders is assigned to the customer, which implies that the amount of the resource occupied is given by sI^+ . Another practical situation is that a customer pays for resource upon arrival, regardless of whether or not the customer is satisfied immediately, or a unit of goods is assigned. For such a system, the resource occupied is $\sum_{m=1}^M s_m I_m$ rather than $\sum_{m=1}^M s_m I_m^+$, and the resource shortage cost is given by $\left(\sum_{m=1}^M s_m I_m - W\right)^+$ rather than $\left(\sum_{m=1}^M s_m I_m^+ - W\right)^+$. It can be shown that all analytical results in this paper are also valid for such a case.

Moreover, other than continuous review (r, Q) policy, studies for limited sharable resource in models with deteriorating items, permissible delay in payment, even periodic review (s, S) policy, can be future directions.

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Appendix

To investigate the cost function $C(\mathbf{r}, \mathbf{Q})$ defined in equation (3.1), the stochastically larger order is used (see, e.g., Ross [11]).

Definition A.1 i) Random variable I_m^1 is stochastically larger than random variable I_m^2 , written as

$$I_m^1 \geq_{st} I_m^2, \text{ if } \Pr\{I_m^1 > a\} \geq \Pr\{I_m^2 > a\} \text{ for all } a;$$

ii) Random vector $\mathbf{I}^1 = (I_1^1, \dots, I_M^1)$ is stochastically larger than random vector $\mathbf{I}^2 = (I_1^2, \dots, I_M^2)$,

written as $\mathbf{I}^1 \geq_{st} \mathbf{I}^2$, if $I_m^1 \geq_{st} I_m^2$ for all $m = 1, \dots, M$.

The following result can be easily shown, and we omit its proof.

Lemma A.1 For I_m^a uniformly distributed on $\{r_m^a+1, r_m^a+2, \dots, r_m^a+Q_m^a\}$ and I_m^b uniformly distributed on $\{r_m^b+1, r_m^b+2, \dots, r_m^b+Q_m^b\}$, $I_m^a \geq_{st} I_m^b$ if and only if $r_m^a \geq r_m^b$ and $r_m^a + Q_m^a \geq r_m^b + Q_m^b$.

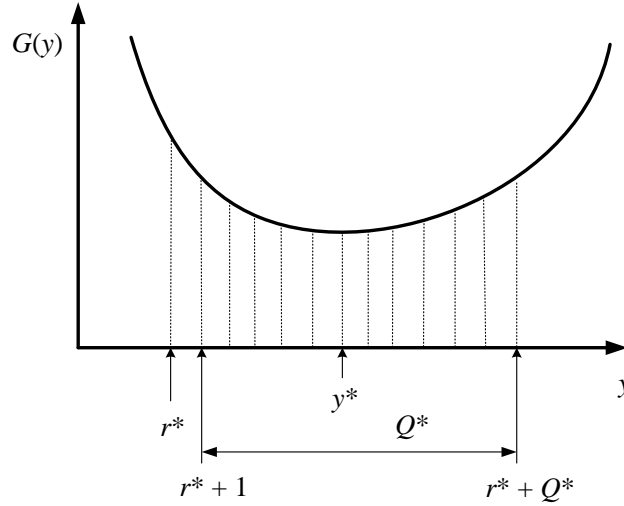


Figure A.1 The function $G(y)$

The basis of the proof is Algorithm 2.1, in which $-G(y)$ is unimodal as shown in Figure A.1. The final resultant policy produced by the algorithm, denoted as (r^*, Q^*) , is the optimal policy of Problem 2.2 with minimal cost $C(r^*, Q^*)$. Eventually, the Q^* smallest values of $G(y)$, i.e., $G(r^*+1), \dots, G(r^*+Q^*)$, are used in the optimization procedure.

According to the procedure of the algorithm, it is easy to see that the following observations hold.

Lemma A.2

- (i) $r^*+1 \leq y^* \leq r^*+Q^*$;
- (ii) $G(y) < G(y')$ holds for any integer $y \in \{r^*+1, r^*+2, \dots, r^*+Q^*\}$ and any integer $y' \notin \{r^*+1, r^*+2, \dots, r^*+Q^*\}$;
- (iii) For any integer y , $G(y) \geq C(r^*, Q^*)$ if and only if $y \notin \{r^*+1, r^*+2, \dots, r^*+Q^*\}$.

Proof of Proposition 4.1

According to the definition of (4.1), for item m , given $(\mathbf{r}^\zeta, \mathbf{Q}^\zeta)$ corresponding to \mathbf{I}^ζ and $(\mathbf{r}^\beta, \mathbf{Q}^\beta)$ corresponding to \mathbf{I}^β , we have

$$f_m \left[(r_m^a, Q_m^a) \middle| (\mathbf{r}^\zeta, \mathbf{Q}^\zeta) \right] = \frac{K_m \lambda_m}{Q_m^a} + \frac{1}{Q_m^a} \sum_{y=r_m^a+1}^{r_m^a+Q_m^a} G_m^\zeta(y), \quad (\text{A.1})$$

$$f_m \left[(r_m^b, Q_m^b) \middle| (\mathbf{r}^\beta, \mathbf{Q}^\beta) \right] = \frac{K_m \lambda_m}{Q_m^b} + \frac{1}{Q_m^b} \sum_{y=r_m^b+1}^{r_m^b+Q_m^b} G_m^\beta(y), \quad (\text{A.2})$$

where

$$G_m^\zeta(y) = g_m(y) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m y^+ - W \right)^+, \quad (\text{A.3})$$

$$G_m^\beta(y) = g_m(y) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m y^+ - W \right)^+. \quad (\text{A.4})$$

To prove $(\mathbf{r}^a, \mathbf{Q}^a) \leq (\mathbf{r}^b, \mathbf{Q}^b)$, by Definition 4.1, it is sufficient to verify that $r_m^a \leq r_m^b$ and $r_m^a + Q_m^a \leq r_m^b + Q_m^b$ for all items $m = 1, \dots, M$.

(1) Proof of $r_m^a \leq r_m^b$

Since $(\mathbf{r}^a, \mathbf{Q}^a)$ is the optimal solution of $\min_{(\mathbf{r}, \mathbf{Q}) \in \Omega^M} F[(\mathbf{r}, \mathbf{Q}) | (\mathbf{r}^\zeta, \mathbf{Q}^\zeta)]$, according to

Lemma A.2, we must have $G_m^\zeta(r_m^a) \geq f_m[(r_m^a, Q_m^a) | (\mathbf{r}^\zeta, \mathbf{Q}^\zeta)]$, which leads to

$$\begin{aligned} & g_m(r_m^a) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m (r_m^a)^+ - W \right)^+ \\ & \geq \frac{K_m \lambda_m}{Q_m^a} + \frac{1}{Q_m^a} \sum_{y=r_m^a+1}^{r_m^a+Q_m^a} \left[g_m(y) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m y^+ - W \right)^+ \right]. \end{aligned} \quad (\text{A.5})$$

It is clear that $(\mathbf{r}^\zeta, \mathbf{Q}^\zeta) \geq (\mathbf{r}^\beta, \mathbf{Q}^\beta)$ or $\mathbf{I}^\zeta \geq_{st} \mathbf{I}^\beta$ implies $\sum_{m \neq k}^M s_m \left(I_m^\zeta \right)^+ \geq_{st} \sum_{m \neq k}^M s_m \left(I_m^\beta \right)^+$.

It can then be shown that $E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m y^+ - W \right)^+ - E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m y^+ - W \right)^+$ is

non-decreasing with respect to y , which is a special case of Lemma A.3 (in the proof of Lemma 4.2).

Therefore, we have

$$\begin{aligned} & Q_m^a \left[E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m \left(r_m^a \right)^+ - W \right)^+ - E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m \left(r_m^a \right)^+ - W \right)^+ \right] \\ & \leq \sum_{y=r_m^a+1}^{r_m^a+Q_m^a} \left[E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_m y^+ - W \right)^+ - E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m y^+ - W \right)^+ \right]. \end{aligned} \quad (\text{A.6})$$

Combining (A.5) and (A.6), we have

$$\begin{aligned} & g_m \left(r_m^a \right)^+ + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m \left(r_m^a \right)^+ - W \right)^+ \\ & \geq \frac{K_m \lambda_m}{Q_m^a} + \frac{1}{Q_m^a} \sum_{y=r_m^a+1}^{r_m^a+Q_m^a} \left[g_m(y) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\beta \right)^+ + s_m y^+ - W \right)^+ \right], \end{aligned} \quad (\text{A.7})$$

which is just $G_m^\beta(r_m^a) \geq f_m \left[\left(r_m^a, Q_m^a \right) \middle| \left(\mathbf{r}^\beta, \mathbf{Q}^\beta \right) \right]$ by (A.1) ~ (A.4). Since $(\mathbf{r}^b, \mathbf{Q}^b)$ is the optimal solution of $\min_{(\mathbf{r}, \mathbf{Q}) \in \Omega^M} F \left[(\mathbf{r}, \mathbf{Q}) \middle| \left(\mathbf{r}^\beta, \mathbf{Q}^\beta \right) \right]$, we have

$$G_m^\beta(r_m^a) \geq f_m \left[\left(r_m^a, Q_m^a \right) \middle| \left(\mathbf{r}^\beta, \mathbf{Q}^\beta \right) \right] \geq \min_{(r_m, Q_m) \in \Omega_m} f_m \left[(r_m, Q_m) \middle| \left(\mathbf{r}^\beta, \mathbf{Q}^\beta \right) \right] = f_m \left[\left(r_m^b, Q_m^b \right) \middle| \left(\mathbf{r}^\beta, \mathbf{Q}^\beta \right) \right],$$

which implies $r_m^a \leq r_m^b$ or $r_m^a \geq r_m^b + Q_m^b + 1$ by Lemma A.2.

Now we prove that $r_m^a \geq r_m^b + Q_m^b + 1$ does not hold, and thus only $r_m^a \leq r_m^b$ is possible. Since $(\mathbf{r}^a, \mathbf{Q}^a)$ is the optimal solution of $\min_{(\mathbf{r}, \mathbf{Q}) \in \Omega^M} F \left[(\mathbf{r}, \mathbf{Q}) \middle| \left(\mathbf{r}^\zeta, \mathbf{Q}^\zeta \right) \right]$, according to Lemma A.2, we must have $G_m^\zeta(r_m^a) > G_m^\zeta(r_m^a + Q_m^a)$, which can be rewritten as

$$\begin{aligned} & g_m(r_m^a) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_k \left(r_m^a \right)^+ - W \right)^+ \\ & > g_m(r_m^a + Q_m^a) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^\zeta \right)^+ + s_k \left(r_m^a + Q_m^a \right)^+ - W \right)^+. \end{aligned} \quad (\text{A.8})$$

By using Lemma A.3, the above implies

$$\begin{aligned}
& g_m(r_m^a) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m (r_m^a)^+ - W \right]^+ \\
& > g_m(r_m^a + Q_m^a) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m (r_m^a + Q_m^a)^+ - W \right]^+,
\end{aligned} \tag{A.9}$$

which is just $G_m^\beta(r_m^a) > G_m^\beta(r_m^a + Q_m^a)$ by (A.1) ~ (A.4). Since $-G_m^\beta(y)$ is unimodal, we have $r_m^a < y_m^*$, where y_m^* is the minimizer of $G_m^\beta(y)$ with $r_m^b + 1 \leq y_m^* \leq r_m^b + Q_m^b$. Therefore, it is impossible to have $r_m^a \geq r_m^b + Q_m^b + 1$.

(2) Proof of $r_m^a + Q_m^a \leq r_m^b + Q_m^b$

The proof of this part is similar to that of Part (1). According to Lemma A.2, we must have

$$G_m^\beta(r_m^b + Q_m^b + 1) \geq f_m \left[(r_m^b, Q_m^b) \middle| (r^\beta, Q^\beta) \right], \text{ which leads to}$$

$$\begin{aligned}
& g_m(r_m^b + Q_m^b + 1) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m (r_m^b + Q_m^b + 1)^+ - W \right]^+ \\
& \geq \frac{K_m \lambda_m}{Q_m^b} + \frac{1}{Q_m^b} \sum_{y=r_m^b+1}^{r_m^b+Q_m^b} \left[g_m(y) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m y^+ - W \right]^+ \right].
\end{aligned}$$

The non-decreasing property of

$$E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\zeta)^+ + s_m y^+ - W \right]^+ - E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m y^+ - W \right]^+ \text{ implies that}$$

$$\begin{aligned}
& Q_m^b \left\{ E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\zeta)^+ + s_m (r_m^b + Q_m^b + 1)^+ - W \right]^+ - E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m (r_m^b + Q_m^b + 1)^+ - W \right]^+ \right\} \\
& \geq \sum_{y=r_m^b+1}^{r_m^b+Q_m^b} \left\{ E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\zeta)^+ + s_m y^+ - W \right]^+ - E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\beta)^+ + s_m y^+ - W \right]^+ \right\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& g_m(r_m^b + Q_m^b + 1) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\zeta)^+ + s_m (r_m^b + Q_m^b + 1)^+ - W \right]^+ \\
& \geq \frac{K_m \lambda_m}{Q_m^b} + \frac{1}{Q_m^b} \sum_{y=r_m^b+1}^{r_m^b+Q_m^b} \left\{ g_m(y) + E \left[\sum_{\substack{k=1 \\ k \neq m}}^M s_k (I_k^\zeta)^+ + s_m y^+ - W \right]^+ \right\},
\end{aligned}$$

which implies $G_m^\zeta(r_m^b + Q_m^b + 1) \geq f_m(r_m^b, Q_m^b) | (r^\zeta, Q^\zeta)$. Since

$$\begin{aligned}
G_m^\zeta(r_m^b + Q_m^b + 1) & \geq f_m(r_m^b, Q_m^b) | (r^\zeta, Q^\zeta) \\
& \geq \min_{(r_m, Q_m) \in \Omega} f_m(r_m, Q_m) | (r^\zeta, Q^\zeta) = f_m(r_m^a, Q_m^a) | (r^\zeta, Q^\zeta),
\end{aligned}$$

we must have $r_m^b + Q_m^b + 1 \geq r_m^a + Q_m^a + 1$ or $r_m^b + Q_m^b + 1 \leq r_m^a$ by Lemma A.2. But in the Proof of Part (1), it has been shown that $r_m^a \geq r_m^b + Q_m^b + 1$ does not hold, thus we have $r_m^a + Q_m^a \leq r_m^b + Q_m^b$.

Therefore, we must have $r_m^a \leq r_m^b$ and $r_m^a + Q_m^a \leq r_m^b + Q_m^b$ for all $m = 1, \dots, M$, which is equivalent to $(r^a, Q^a) \leq (r^b, Q^b)$. This completes the proof of Proposition 4.1.

Proof of Lemma 4.2

To show Lemma 4.2, the following lemma is proved first.

Lemma A.3 For the following difference function

$$\delta = E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k I_k^+ + s_m (I_m^1)^+ - W \right)^+ - E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k I_k^+ + s_m (I_m^2)^+ - W \right)^+,$$

if $I_m^1 \geq_{st} I_m^2$, then δ is non-decreasing as \mathbf{I} increases in the sense of the stochastically larger order.

Proof of Lemma A.3. Define the following function

$$f(x) = E \left(x^+ + s_m (I_m^1)^+ - W \right)^+ - E \left(x^+ + s_m (I_m^2)^+ - W \right)^+.$$

For $x_1 \geq x_2$, we consider the difference $f(x_1) - f(x_2)$, which is given by

$$\begin{aligned}
& f(x_1) - f(x_2) \\
&= E \left[\left(x_1^+ + s_m (I_m^1)^+ - W \right)^+ - \left(x_2^+ + s_m (I_m^1)^+ - W \right)^+ \right] \\
&\quad - E \left[\left(x_1^+ + s_m (I_m^2)^+ - W \right)^+ - \left(x_2^+ + s_m (I_m^2)^+ - W \right)^+ \right].
\end{aligned}$$

If $x_1 \geq x_2$, then $\left(x_1^+ + y^+ - W \right)^+ - \left(x_2^+ + y^+ - W \right)^+$ is a non-decreasing function with respect to y .

It is known in the literature (see, e.g., Ross (1996)) that if $I^1 \geq_{st} I^2$, then $E[g(I^1)] \geq E[g(I^2)]$ for any non-decreasing function $g(x)$. Therefore, if $I_m^1 \geq_{st} I_m^2$, then

$$\begin{aligned}
& E \left[\left(x_1^+ + s_m (I_m^1)^+ - W \right)^+ - \left(x_2^+ + s_m (I_m^1)^+ - W \right)^+ \right] \\
&\geq E \left[\left(x_1^+ + s_m (I_m^2)^+ - W \right)^+ - \left(x_2^+ + s_m (I_m^2)^+ - W \right)^+ \right].
\end{aligned}$$

Thus, we have $f(x_1) - f(x_2) \geq 0$, i.e., $f(x)$ is a non-decreasing function with respect to x . Consequently, we have $\delta \geq 0$. Lemma A.3 is proved.

Now, we go back to the proof of Lemma 4.2. Let I^0 correspond to (r^0, Q^0) and I correspond to (r, Q) . For $(r, Q) \leq (r^0, Q^0)$, it follows that

$$\begin{aligned}
C(r, Q) &= \sum_{m=1}^M c_m(r_m, Q_m) + E \left(\sum_{k=1}^M s_k I_k^+ - W \right)^+ \\
&= \sum_{m=1}^M c_m(r_m, Q_m) + E \left(\sum_{k=1}^M s_k (I_k^0)^+ - W \right)^+ \\
&\quad - \left[E \left(\sum_{k=1}^M s_k (I_k^0)^+ - W \right)^+ - E \left(\sum_{k=1}^M s_k I_k^+ - W \right)^+ \right] \\
&= \sum_{m=1}^M c_m(r_m, Q_m) + E \left(\sum_{k=1}^M s_k (I_k^0)^+ - W \right)^+ \\
&\quad - \left[E \left(\sum_{k=1}^M s_k (I_k^0)^+ - W \right)^+ - E \left(s_1 I_1^+ + \sum_{k=2}^M s_k (I_k^0)^+ - W \right)^+ \right]
\end{aligned}$$

$$\begin{aligned}
& - \left[E \left(s_1 I_1^+ + \sum_{k=2}^M s_k \left(I_k^0 \right)^+ - W \right)^+ - E \left(s_1 I_1^+ + s_2 I_2^+ + \sum_{k=3}^M s_k \left(I_k^0 \right)^+ - W \right)^+ \right] \\
& - \left[E \left(s_1 I_1^+ + s_2 I_2^+ + \sum_{k=3}^M s_k \left(I_k^0 \right)^+ - W \right)^+ - E \left(s_1 I_1^+ + s_2 I_2^+ + s_3 I_3^+ + \sum_{k=4}^M s_k \left(I_k^0 \right)^+ - W \right)^+ \right] \\
& \vdots \\
& - \left[E \left(s_1 I_1^+ + \cdots + s_{M-1} I_{M-1}^+ + s_M \left(I_M^0 \right)^+ - W \right)^+ - E \left(\sum_{k=1}^M s_k I_k^+ - W \right)^+ \right].
\end{aligned}$$

By Lemma A.3, we obtain

$$\begin{aligned}
C(\mathbf{r}, \mathbf{Q}) & \geq \sum_{m=1}^M c_m(r_m, Q_m) + E \left(\sum_{k=1}^M s_k \left(I_k^0 \right)^+ - W \right)^+ \\
& - \left[E \left(\sum_{k=1}^M s_k \left(I_k^0 \right)^+ - W \right)^+ - E \left(\sum_{k=2}^M s_k \left(I_k^0 \right)^+ + s_1 I_1^+ - W \right)^+ \right] \\
& - \left[E \left(\sum_{k=1}^M s_k \left(I_k^0 \right)^+ - W \right)^+ - E \left(\sum_{\substack{k=1 \\ k \neq 2}}^M s_k \left(I_k^0 \right)^+ + s_2 I_2^+ - W \right)^+ \right] \\
& \vdots \\
& - \left[E \left(\sum_{k=1}^M s_k \left(I_k^0 \right)^+ - W \right)^+ - E \left(\sum_{k=1}^{M-1} s_k \left(I_k^0 \right)^+ + s_M I_M^+ - W \right)^+ \right] \\
& = \sum_{m=1}^M \left[c_m(r_m, Q_m) + E \left(\sum_{\substack{k=1 \\ k \neq m}}^M s_k \left(I_k^0 \right)^+ + s_m I_m^+ - W \right)^+ \right] + (1-M) E \left(\sum_{k=1}^M s_k \left(I_k^0 \right)^+ - W \right)^+ \\
& = F \left[(\mathbf{r}, \mathbf{Q}) \middle| (\mathbf{r}^0, \mathbf{Q}^0) \right].
\end{aligned}$$

The proof is completed.