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MATHEMATICAL EXPERIMENTS: A NEW-DESIGNED COURSE FOR NON-MATHEMATICAL UNDERGRADUATES IN CHINESE UNIVERSITIES

Jinxing Xie

Tsinghua University

jxie@math.tsinghua.edu.cn

More and more universities in China are offering mathematical experiments, a new-designed course, to undergraduates not majoring in mathematics. The primary idea of the course is to activate the students to learn mathematics by doing mathematics themselves, and to cultivate the students' mathematical application consciousness and ability. However, different understandings exist on the issues related to what kind of mathematics should be taught, and what kind of pedagogies should be accepted for the course. This paper gives a brief introduction to the current status of the course in China, and shares our experiences in designing and teaching the course at Tsinghua University, in which the course tries to integrate the problem-solving methods such as mathematical modeling and mathematical software, and the commonly-used mathematical methods such as numerical computing, optimization and statistics. An example experiment in the course is also presented.
Mathematical experiments; Mathematical Modeling; Mathematical software; Problem solving; Course design.

INTRODUCTION

Supported by Ministry of Education of China, a project entitled "Reforms on the Course System and Teaching Content of Higher Mathematics (For Non-Mathematical Specialties)" commenced in China starting from 1995. Several years later, the research report of the project was released (Xiao 2000, Xiao 2002), which has been producing profound impacts on the understanding of the role of university mathematics education, and has been significantly changing the situation of mathematics education in Chinese universities.

One of the main contributions of the project lies in that four basic mathematical courses – Calculus, Algebra & Geometry, Random Mathematics, and Mathematical Experiments – are proposed to all undergraduates of non-mathematical majors. Depending on students' different backgrounds and interests, the course Calculus is suggested to have 60-120 lectures (each lecture usually lasts for 45 minutes in China), while about 60-90, 30-50 and 30-50 lectures are suggested for the latter three courses respectively (Xiao 2000). The first three courses are traditional ones for students' basic mathematical training, but are redesigned to reflect the modernization of the mathematical knowledge and the needs for students' whole quality cultivation. However, the course Mathematical Experiments is a completely new-designed one, with the aim to cultivate the students' mathematical application consciousness and ability, by making use of the mathematical experimentation on computers.

Guided by this project, in the last decade, more than 220 universities in China started to offer the Mathematical Experiments course to undergraduates, and in order to facilitate the teaching of the course, established their Laboratories of Mathematical Experiments (Xie, 2012). However, different understandings exist on the issues related to what kind of mathematics should be taught in the course, and what kind of pedagogies should be accepted for the course, thus there is far from convergence concerning the specific contents of the course. In this paper, we give a brief introduction to the current status of the course in China, and share our experiences in designing and teaching the Mathematical Experiments course at Tsinghua University.

CURRENT STATUS OF THE MATHEMATICAL EXPERIMENTS COURSE IN CHINA

It is noticed that the concept of experiments in mathematics is nothing new, since mathematicians have always done this, with paper and pencil before, and nowadays using computers to investigate a large number of cases, or perform computations that are difficult to do by hand. As early as in 1982, Grenander (1982) published a book on mathematical experiments. More often, Mathematical Experiments might be called "Experimental Mathematics". There is even an international journal, Experimental Mathematics, starting from 1992, devoted entirely to it. Weisstein (2012) expresses “Experimental Mathematics as a type of mathematical investigation in which computation is used to investigate mathematical structures and identify their fundamental properties and patterns”. Similar opinions are also proposed in Cobb et al. (1997), Borwein and Bailey (2003), Borwein et al. (2004), and Li et al. (2003). For example, Borwein and Bailey (2003, pp. 2-3) use the term "experimental mathematics" to mean the methodology of doing mathematics that includes the use of computation for:

> (1) Gaining insight and intuition; (2) Discovering new patterns and relationships; (3) Using graphical displays to suggest underlying mathematical principles; (4) Testing and especially falsifying conjectures; (5) Exploring a possible result to see if it is worth formal proof; (6) Suggesting approaches for a formal proof; (7) Replacing lengthy hand derivations with computer-based derivations; (8) Confirming analytically derived results.

We may call them the eight primary functions of computation in mathematics. According to this understanding, almost all the mathematical contents can be included into the Mathematical Experiments course. Specifically, there are mainly four forms of Mathematical Experiments courses in China universities up to now. We briefly summarize their characteristics in below.

- The first form of the course is based on applications of computers in the first three basic courses (Calculus, Algebra and Geometry, Random Mathematics). In our opinion, the so-called Mathematical Experiments courses in this form at some universities in China are essentially developing CAI (Computer-Aided Instructions) technologies in the first three basic courses. For example, the teachers may use computers in their lectures to enhance the visibility for some mathematical concepts, or to provide some intuition for some mathematical theorems.

- The second form of the course is based on applications of computers in the traditional Mathematical Modeling course, which is already very popular in Chinese universities (According to Xie (2012), more than 1100 universities – about 50% of all universities in China – are offering this course to undergraduates up to now). The traditional modeling courses in China are usually theoretically oriented and taught by introducing modeling cases, but computers and mathematical software are seldom used. In our opinion, the course is still a Mathematical Modeling course even when applications of computers in the course are getting more and more popular.
- The third form of the course is to use computers to investigate mathematical structures and identify their fundamental properties and patterns. This type of the course puts more attention on enhancing students' experiences in mathematical research and mathematical discoveries. The experiences in doing mathematics this way are certainly important to all students. However, currently in China, the mathematical knowledge selected for the course in this form is more mathematics-oriented, and is probably more suitable to the students who like to become a mathematician in the future, but perhaps it does not fit all the students, especially the students not majoring in mathematics.
- The fourth form of the course tries to integrate the problem-solving methods such as mathematical modeling and mathematical software, and the common-used mathematical methods such as numerical computing, optimization and statistics. The course not only emphasizes the need for students to learn applied mathematics (in particular, numerical computing, optimization and statistics) by doing mathematics themselves, but also focuses on cultivating the students' mathematical application consciousness and ability. We think this type of mathematical experiments course is very suitable to the students majoring in non-mathematical specialties, and thus it is adopted in our university (Tsinghua University) and some other universities in China. We will give more details in next section.

MATHEMATICAL EXPERIMENTS COURSE AT TSINGHUA

Aim and scope

As we have mentioned, in China, mathematical experiments course is the last one of the four basic mathematical courses for undergraduate students of non-mathematical majors. When students finish these four courses, they may have no chance to learn more mathematical courses before they graduate. Therefore, we think it is important for the students in the last course (i.e., Mathematical Experiments) to learn some mathematics most commonly used in practice. It is not strange that one may argue that too many mathematical contents can be included, since almost all mathematics could be applicable. But at Tsinghua University, the course Mathematical Experiments has only 45 lectures with each consisting of 45 minutes, thus we should select the course contents very carefully.

The mathematical experiments course at Tsinghua tries to integrate, expect the three basic courses above mentioned, the most fundamental contents of several traditional courses such as Mathematical Modeling, Mathematical Software, Numerical Analysis, Optimization, and Statistics, at least partially. We think it is crucial for the undergraduate students with non-mathematical majors to know the basic concepts and techniques in these fields, which are usually only taught to mathematics students before. Therefore, we position the course on teaching and learning applied mathematics (in particular, numerical computing, optimization and statistics) by the means of mathematical experiments with mathematical software.

Specifically, as preparations to the course, two experiments on mathematical modeling and software are first conducted (we choose MATLAB and LINGO as experimental tools in our course, which are mathematical software products from The MathWorks and LINDO Systems respectively; for more information about these two products, please refer to their homepages at http://www.mathworks.com and http://www.lindo.com respectively).

Then the course consists of 12 experiments (Jiang, 2001; Jiang et al., 2010) as below:

- Four of them are on numerical computing: Interpolations and numerical integrations; Numerical solutions for ODE (Ordinary Differential Equations); Numerical solutions for the system of linear equations; Numerical solutions for the system of nonlinear equations.
- Four of them are on optimizations: Unconstrained optimizations; Linear programming; Nonlinear programming; Integer programming.
- Four of them are on statistics: Data collecting and statistics description; Parameter estimations and hypothesis tests; Multi-variable regressions; Simulation.

Implementation

Popularization of mathematical software on personnel computers enhances students' numerical computing functions and image processing functions, which provides technical feasibility and ensures the students' learning efficiency for such a course. In every experiment, the lecture starts from modeling the simplified real problems, followed by mathematical knowledge (basic computing algorithms in computing, optimization and statistics, and their software implementations), and ended with the solutions and verifications of the real problems introduced in the beginning of the lectures. Complicated mathematical theorems are usually not proved in a mathematically rigorous way, but are intuitively shown based on the observations from the computations with mathematical software, making use of the above-mentioned eight primary functions of computations in mathematics. After the lectures (out of the class), the students must finish their homework, in which they are asked to use computers and the mathematical software to solve some other simplified practical problems, from modeling the problem to obtaining the last solutions of the problems. For each of the assignments, every student must submit his/her final report to the teachers for evaluation. There is also a final exam which is conducted in the university's computer laboratory. The final evaluation for each student is based on both the reports he/she submitted and his/her marks in the final exam.

Responses from students

In 2003, we conducted a questionnaire survey in the students enrolled in the course. 379 responses were received, and the analysis revealed that the course is very successful in the sense that the students like the course very much and they also show great interests to learn more mathematics. Some of the results from the survey are listed in below.

(a) Has the course achieved its objective? Yes (92%) No (8%)

(b) Is the content of the course appropriate? Yes (89%) No (11%)

(c) Do you think the course is helpful to you? Yes (95%) No (5%)

(d) How you do think about the difficulty of the course?

Proper (68%) Too difficult (20%) Too easy (9%) No answer (12%)

(e) How you do think about the burden of the experiments (assignments)?

Proper (74%) Too many (13%) Too little (12%) No answer (1%)

(f) How you do think about the teaching methods used in the class?

Good (67%) Not good (9%) No answer (24%)

AN ILLUSTRATIVE EXPERIMENT

The Problem

In the experiment "Numerical solutions for ODE", after introducing some practical examples which can be modeled by ODE, the teacher presents the basic ideas of numerical solutions for ODE, in particular, the construction and error analyses of the Runge-Kutta algorithm, which is commonly-used to solve ODE. Then the teacher solves the examples introduced at the beginning of the lecture with the software MATLAB (e.g., using the MATLAB command "ode45"). Finally, as one of the assignments to the students, the teacher asks the students to solve the following problem outside the class (This problem is also extensively used in classrooms for other courses, e.g., see Riede (2003)):

> There is a river with width of 100 meters (m), and there is a boat trying to cross the river from one side to the other side. The speed of the water is 1 m/s and the speed of the boat is 2 m/s. Suppose that during the course the head of the boat always points to the destination which is on the right opposite side of the starting point of the boat. Answer the following questions:
>
> (i) What is the route (path) of the boat on the water?
>
> (ii) How long does the boat take in order to cross the river?
>
> (iii) Change the values of the river width, the speeds of the water and the boat, and then answer the same questions as above. Can you find anything interesting from your observations?

The Model

Most of the students can model the problem as an ODE with initial conditions as below (cf. Figure 1, the boat at position P(x, y) wants to cross the river from A to B):

$$\begin{aligned}\frac{dx}{dt} &= v_1 - v_2\cos\alpha \\ \frac{dy}{dt} &= -v_2\sin\alpha\end{aligned} \quad \text{or} \quad \begin{cases}\dfrac{dx}{dt} = v_1 - v_2 x/\sqrt{x^2+y^2} \\ \dfrac{dy}{dt} = -v_2 y/\sqrt{x^2+y^2}\end{cases} \qquad (1)$$

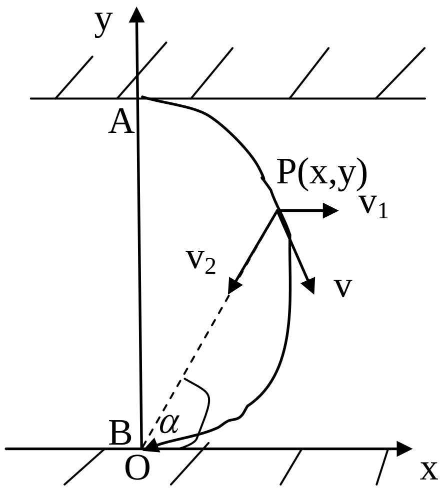

Figure 1: The boat crossing the river

Here v_1 and v_2 are the speeds of the water and the boat, respectively; d is the width of the river; and (x, y) is the position of the boat at time t. The initial conditions for this ODE are $x(0)=0$ and $y(0)=d$.

Question (i)

In order to answer Question (i), one can solve Eqn. (1) numerically and then plot (x, y) in the plane. In fact, in one of my classes, all of the 270 students tried to follow this way (please refer to the next subsection for how to solved Eqn. (1) numerically with MATLAB).

In addition to this, nearly 25% students (67 out of 270) also analytically obtained the relationship between x and y. In fact, it is easy to eliminate the time parameter "t" from (1):

$$\frac{dx}{dy} = -\frac{v_1}{v_2}\sqrt{(\frac{x}{y})^2 + 1} + \frac{x}{y}, \qquad (2)$$

in which x=0 when y=d.

Let $x/y=u$, Eqn. (2) becomes

$$y\frac{du}{dy} = -k\sqrt{1+u^2}, \ \ (k = \frac{v_1}{v_2}), \qquad (3)$$

in which u=0 when y=d.

The solution of Eqn. (3) is $\sqrt{1+u^2} = d^k y^{-k} - u$, which means the solution of (2) is

$$x = \frac{d}{2}\left[\left(\frac{y}{d}\right)^{1-k} - \left(\frac{y}{d}\right)^{1+k}\right], \ \ (k = \frac{v_1}{v_2}). \qquad (4)$$

This is the equation governing the travelling path of the boat on the water. The students then can check whether this analytical solution is consistent with the numerical solution obtained with MATLAB.

Question (ii)

In order to answer Question (ii), the students try to numerically solve Eqn. (1) following the similar process just learnt from the lecture, i.e., using the MATLAB command "ode45". However, the command "ode45" needs the students to input the range for the parameter "*t*", i.e., the staring time (assume to be zero) and the ending time of the movement of the boat. Many students face the difficulty to set the range for this parameter, since they do not know how long the boat would take in order to cross the river.

Usually, the students just set a large enough range for the time parameter, e.g., [0,100]. However, when they run "ode45" under this set of parameters, the MATLAB keeps running and running, and the program seems never stop to reach a solution! A little group of students just stopped here and sent me emails to say "the ode45 command has a serious bug and cannot solve the problem".

However, other students tried to use a try-and-error technique to find the reasonable terminal time for the boat. For example, they may use the bisection method to try other time intervals such as [0, 50] followed by [0, 75], [0, 62.5], …. Finally, they found the terminal time should be 66.7 (seconds). That's to say, when setting the range of the time to [0, 66.7] in "ode45", the program runs and stops normally.

Why does this happen? A small group of students thought of it in-depth. Examination of the model in Eqn. (1) reveals that the formula is mathematically problematic, because when the boat reaches the destination point B(0, 0), we have $x=y=0$ and Eqn. (1) is meaningless. This might be the reason why "ode45" cannot run successfully when a large enough range for the time parameter is set.

Are there any better approaches to deal with the problem? Some students propose a very simple but more systematic approach to avoid the try-and-error procedures above mentioned. They solve the problem by "ode45" step by step, i.e., in each step the boat proceeds from the current position in time t to the next position in a very near future time, e.g., $t+0.1$. Whenever the boat reaches a new position, check whether the destination is reached. If yes, the procedure stops; otherwise, go to the next position. This trick can deal with the problem successfully!

More clever approaches can be found from the students' reports for this assignment. Noticing that in the original model (1), the direction of the boat is changed after it cross over the destination point, but this is unreasonable. Thus some students just change the second equation in (1) from $dy/dt = -v_2 y/\sqrt{x^2+y^2}$ to $dy/dt = -v_2 |y|/\sqrt{x^2+y^2}$, and then "ode45" works normally. Of course, we are only interested the solution in the time interval [0, 66.7], thus we should discard the solution outside of this range.

Some students tried another approach – changing model (1) to the following equivalent one:

$$\frac{dx}{dy} = \frac{v_1 - v_2 x/\sqrt{x^2+y^2}}{-v_2 y/\sqrt{x^2+y^2}}, \qquad \frac{dt}{dy} = \frac{1}{-v_2 y/\sqrt{x^2+y^2}}, \tag{5}$$

with initial conditions $x(d)=0$ and $t(d)=0$.

Clearly, for this ODE, the starting value *y* for is *d* and the ending value is 0. Therefore, there are no difficulties to set the parameters for integration with "ode45". In fact, integrating (5) for *y* over [*d*, 0] using "ode45", the solution can be successfully found. (When one uses the integration interval [*d*, 0], MATLAB may still has some difficulties depending on its different versions. However, one can use the integration interval [*d*, *e*], where *e* is relatively small positive value close to 0, e.g. *e* =0.01).

Question (iii)

This task can be easily done after the tasks (i) and (ii) are well done.

An Alternative Model

To my surprise, a very smart alternative approach is also suggested by a couple of students. They use the water as the reference point (cf. Figure 2), therefore at time *t* the boat is at position Q(d, v_1t) when the boat is at position P(x,y). They model the problem as

$$\begin{cases} \dfrac{dx}{dt} = v_2 \cos\alpha \ , \\ \dfrac{dy}{dt} = v_2 \sin\alpha \ , \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{dx}{dt} = \dfrac{v_2(d-x)}{\sqrt{(d-x)^2 + (v_1t - y)^2}} \ , \\ \dfrac{dy}{dt} = \dfrac{v_2(v_1t - y)}{\sqrt{(d-x)^2 + (v_1t - y)^2}} , \end{cases} \qquad (6)$$

with initial conditions $x(0)=y(0)=0$.

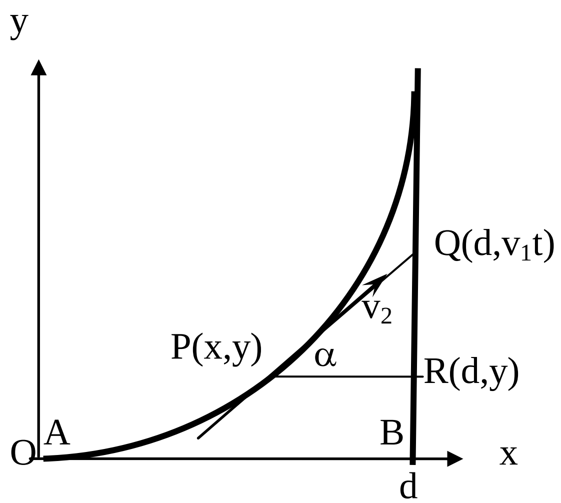

Figure 2: An alternative modelling approach

Eqn. (6) can be solved by MATLAB (e.g. "ode45" command) directly for any integration range for the parameter "*t*". Besides, similarly to the model (1), one can also analytically obtain the relationship between *x* and *y* from the model (6). In fact, from (6) we have

$$\frac{dy}{dx} = \frac{v_1 t - y}{d - x}. \qquad (7)$$

Therefore

$$(d-x)\frac{d^2y}{dx^2}=v_1\frac{dt}{dx}. \qquad (8)$$

Denote $ds=\sqrt{(dx)^2+(dy)^2}$. Noticing $ds/dt=v_2$, Eqn. (8) can be rewritten as

$$(d-x)\frac{d^2y}{dx^2}=\frac{v_1}{v_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \qquad (9)$$

Let $dy/dx=u$ and $k=v_1/v_2$, Eqn. (9) becomes

$$(d-x)\frac{dp}{dx}=k\sqrt{1+p^2}, \text{ with } p(0)=0. \qquad (10)$$

The solution of Eqn. (10) is $\sqrt{1+p^2}+p=(1-x/d)^{-k}$, which means $dy/dx=p=[(1-x/d)^{-k}-(1-x/d)^{k}]/2.$ Therefore, the solution of Eqn. (9) is

$$y=\frac{d}{2}\left[\frac{1}{1+k}\left(\frac{d-x}{d}\right)^{1+k}-\frac{1}{1-k}\left(\frac{d-x}{d}\right)^{1-k}\right]+\frac{kd}{1-k^2}. \qquad (11)$$

This is the equation governing the travelling path of the boat on the water. It is interesting that one can also determine the time the boat takes to cross the river from (11). When the boat reaches the terminal, $x=d$. Therefore from (11), we have $y=\frac{kd}{1-k^2}$. Thus the time needed for the boat to cross the river should be $\frac{kd}{1-k^2}\Big/v_1=\frac{v_2d}{v_2{}^2-v_1{}^2}$. For $v_1=1$, $v_2=2$ and $d=100$, this is just 200/3, or about 66.7 (seconds), which is consistent with the results mentioned above.

Final Comments

As a teacher, I'm deeply impressed by abovementioned various approaches suggested by the students in their experimental reports. I also noticed that some students go further to investigate how the boat can cross the river in the shortest time. That is, along which kind of paths, the boat can reach the destination most quickly. Of course this is a problem interested by many students, and it can be modelled as a problem of calculus of variations. I omit the discussion about that here since it is beyond the common knowledge of the mathematical background of usual undergraduate students majoring in non-mathematical specialties.

After reading the aforementioned variants of approaches to answer the questions in the students' experimental reports, I'm really impressed by the students' creativity and achievements!

SUMMARY

In this paper, we have shared our experiences in designing and teaching the Mathematical Experiments course at Tsinghua University. The objective of the course is to teach non-mathematics students to learn applied mathematics (in particular, numerical computing, optimization and statistics) by the means of mathematical experiments, mathematical modeling and mathematical software. The course was offered starting from 1998, as an optional course in the first several years for non-mathematics students and now as a must-select course for science and engineering students. The course is very successful in the sense that the students like the course very much and they also show great interests to learn more mathematics.

Although we are trying to integrate mathematical modeling, mathematical software, numerical analysis, optimization and statistics into a single course of mathematical experiments, we also realize that this work is difficult. We should not simply mix these traditional courses together, or just treat the course as a mathematical software course. Furthermore, as modern mathematics is changing from time to time, it is challenging to revise and redesign the course to reflect the advances in mathematics and mathematical software.

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