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### MATHEMATICAL MODELLING MODULES FOR CALCULUS TEACHING

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**Abstract**-*This paper introduces a two-year project on incorporating ideas and methods of mathematical modelling into the teaching of main mathematical courses in Chinese universities and colleges, initiated by the National Organizing Committee (NOC) of the China Undergraduate Mathematical Contest in Modelling (CUMCM) in 2003. The importance and necessity of the project are briefly discussed. The project's emphases are put on designing mathematical modelling modules, which include the whole mathematical modelling process for solving real-world problems and should be easily understood and can be effectively used for the existing courses. The use of the modules will not disturb instructors' regular teaching, but will stimulate and raise students' interest in studying mathematics. In particular, recommendations on how and where this can be done in the existing calculus teaching are discussed. A sample module "Why a Coca Cola can takes such a shape" is presented in detail.* 

### **1. INTRODUCTION**

More and more professors and administrators from universities and the Ministry of Education in China realize that the teaching of mathematics at university level is very important. Especially, they realize that mathematical modelling (MM) techniques are very important from the very success of the China Undergraduate Mathematical Contest in Modelling (CUMCM). Mathematical modelling and associated computations and simulation are becoming critical tools in the engineering design process. Scientists and engineers rely increasingly on computational methods and must have sufficient experience in mathematical / computational methods to be able to choose the correct methods and interpret the accuracy and reliability of the results (Friedman et al., 1992). Modelling now permeates the daily lives of professionals. Mathematical modelling has become indispensable as a research tool, particularly in connection with the appearance of computers. It allows one to design new techniques to find optimal regimes for the solutions of complex scientific problems and predict new phenomena. Therefore, the mathematical education of our future engineers and scientists needs to change to reflect this new reality, in particular through its early introduction at university level, where it has become desirable, or even mandatory (Friedman et al., 1992; Hazewinkel, 1995).

In China, more than 40,000 students participate in CUMCM and similar competitions in many universities each year. They have learned a lot of MM and shown competence in their subsequent courses, projects, and later careers. We have about five million students entering into universities and colleges each year; most of them have to study calculus for at least one or two semesters. Many do not understand why they have to spend so much time in studying calculus or other mathematical courses, and why it is important for their future careers, therefore their study lacks motivation and initiative. In order to solve these kinds of problems, the NOC of CUMCM initiated a two-year project in 2003 titled "incorporating ideas and methods of mathematical modelling into the main mathematical courses in universities and colleges" (Ye, 2003). There are three main mathematical courses in universities and colleges: calculus, linear algebra, and elements of probability and statistics. The project's emphases are put on designing and writing feasible modules on MM, which include the explanation of the whole mathematical modelling process from real-world problems, and they can be embedded or effectively used for the teaching of existing courses. Most importantly, the use of the modules will not disturb instructors' regular teaching but will stimulate and raise students' interest in studying mathematics. In particular, recommendations on how and where this can be done in the existing calculus teaching will be discussed in this paper. As an example, a sample module titled "Why a Coca Cola can takes such a shape" is presented in detail

### 2. AIMS OF THE PROJECT

- 2.1 Organizing and encouraging instructors from different institutions to write modules on MM that can be used in the teaching of calculus, linear algebra, and elements of probability and statistics.
- 2.2 Using these modules in the teaching of these three courses in a few institutions for testing and identifying problems. In the teaching we will emphasize not only the "know how", but also the "know why". Our expectation is that students who have learned theses modules should know why MM and mathematics are important and even critical for their careers and for strengthening their creativity and competitive ability. Especially, we hope more students will get interested in mathematics and work harder on it.
- 2.3 Communicating and discussing good modules at seminars for incubating and training instructors from various universities and encouraging them to use these modules in their own teaching. The collection of the best modules will be published.
- 2.4. We hope that the implementation of the project will go well, and more students will be interested in MM and take further courses on MM and related mathematical courses, and as a result, we will have more talented student to participate in the CUMCM.

### **3. PRINCIPLES OF WRITING MM MODULES**

- 3.1 The problem should be a real-world problem and is easy to understand.
- 3.2 The module will embody the whole mathematical modelling process.

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- 3.3 The module should be attractive to instructors who are teaching calculus, linear algebra, and elements of probability and statistics, so that they are willing to read and use these modules in their teaching activities.
- 3.4 The module should meet needs of various students.
- 3.5 Most important is that the use of the modules will not disturb instructors' regular teaching. In general, only less than 2 extra teaching hours is needed for teaching one module.

# 4. A SAMPLE MODULE - WHY A COCA COLA CAN TAKES SUCH A SHAPE?

We present a sample module for calculus teaching in detail in this section. In almost all the calculus courses, there is a section on optimization problems as applications of differentiation. A typical such problem in many calculus texts is as follows.

A cylindrical can is to be made to hold some liters of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can (see, for example, Stewart, 2003). It is essentially a geometric optimal problem, that is, given a right circular cylinder and its volume, find its diameter and height that minimize its surface area. Our comment is that before using our module teachers first solve this simple problem together with their students. This is easy and the solution can be found in most calculus textbooks.

Solution. Denote *r* the radius, *d* the diameter (d = 2r), *h* the height, *V* the volume and *S* the surface area of the right circular cylinder, and *V* is given. Then

$$V = \pi r^2 h, \ h = V / \pi r^2, \ S = 2\pi r h + 2\pi r^2.$$
(1)

Therefore, the optimization problem is

$$\begin{cases} \min_{r>0, h>0} S(r, h) \\ s.t. V = \pi r^2 h \end{cases}$$
(2)

<u>Method 1</u>: Reduce the constrained minimization problem to an unconstrained minimization problem. Substituting h into the expression of S, we have

$$S = 2\pi r V / \pi r^{2} + 2\pi r^{2} = 2(\pi r^{2} + V / r).$$
(3)

Finding critical points as follows:

$$S' = 2(2\pi r - V/r^2) = \frac{2}{r^2}(2\pi r^3 - V) = 0, \ r = \sqrt[3]{\frac{V}{2\pi}}.$$
 (4)

Since  $S'' = 4(\pi + V/r^3) > 0$  for r > 0 and there is only one critical point, the surface area is minimized at  $r = \sqrt[3]{V/2\pi}$ . Thus the optimal height is

$$h = \frac{V}{\pi} \sqrt[3]{\frac{(2\pi)^2}{V^2}} = \sqrt[3]{\frac{(2\pi)^2 V^3}{V^2 \pi^3}} = \sqrt[3]{\frac{2^3 V}{2\pi}} = 2r = d \cdot$$
(5)

It means that the diameter equals the height of the right circular cylinder.

<u>Method 2</u>: Using the arithmetic mean and geometric mean inequality (Isoperimetric Inequality). We will use this method later, where the same conclusion d=h can be obtained.

In our module, we have a sub-section on what is mathematical model and what is mathematical modelling. Its main contents are as follows.

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A mathematical model is a (rough) description of a class of real-world problems or phenomena expressed using mathematical symbolism. The process of building, solving, and validation of it is called mathematical modelling. The methods of mathematical modelling are not new, and almost all the persons from ancient to modern who use mathematical modelling (including some examples such as Euclid (about BC 330 ~ BC 275), Archimedes (about BC 287 ~ BC 212), Galileo, G. (1564 ~ 1642), Navier, C.M.L.H. (1785 ~ 1836), and Newton, I. (1642-1727), etc.).

- Key steps of mathematical modelling (Wan, 1990):
- Step 1: Observation and analysis of the key aspects of the real-world problem.
- Step 2: Simplification and making reasonable assumptions (it is quite difficult).
- Step 3: Determine variables and parameters.
- Step 4: Using certain geometrical relations or physical laws to relate these variables and forming a concrete mathematical problem (we may call it a mathematical model at this stage) (note that the math problem might be very complicated).
- Step 5: Solving this mathematical model analytically or approximately if possible (note that it might be very difficult to solve it).
- Step 6: Validation: for example, by using historical or experimental data to check that if the result is right (reasonable) or not to some extent (it is also not an easy job).
- Step 7: If the results are positive we can use it, otherwise we have to go back to all the previous steps to check if there is something wrong, and if so, correct them, and rerun the whole process again.

The emphasis is put on reasonable assumptions, solving math problems and validation.

**Organizing the teaching.** Before teaching the problem "why a Coca Cola can takes such a shape" (may be before the ending of your last lecture), ask students to review the related text and do the typical example mentioned above, and take a Coca Cola can to measure its dimensions and other data (for instance, approximately the radius of its top cover is 3cm, the radius of its middle part is 3.3cm, and its height is 13cm). The content of the drink is 355 millilitres (ml) (about 355cm<sup>3</sup>), and there are about 10cm<sup>3</sup> in the can remains empty, thus the volume inside the can is 365cm<sup>3</sup>. Then you can teach the MM process step by step together with your students in the classroom when you start your next lecture. You can show students an empty Coca Cola can and let them feel the top cover is much harder and thicker than the other parts of the can. Its central section is quite close to the shape shown in Figure 1(a).

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Figure 1. The shape of the Coca Cola Can.

A simplified model. In this case we have to consider the volume of the cover material. We note "the 'simple-to-elaborate' approach is so important in mathematical modelling" (see Wan, 1990).

Step 2. Simplification. Assume that the can is a right circular cylinder as shown in Figure 1(b).

Step 3. Determining variables and parameters. The radius r and height h of the can are variables, and the volume of the can V is a given known parameter. The thickness of the top cover is  $\alpha b$  and the thickness of the other part is b (cm). Here b and  $\alpha$  are parameters.

Step 4. Building the mathematical model. The volume of the cover material of the can (SV) and the volume inside the can (V) can be calculated as

$$SV(r,h) = h[\pi(r+b)^2 - \pi r^2] + b\pi(r+b)^2 + \alpha b\pi(r+b)^2$$
  
=  $2\pi rhb + (1+\alpha)\pi r^2b + 2(1+\alpha)\pi rb^2 + \pi hb^2 + (1+\alpha)\pi b^3,$  (6)

$$V(r,h) = \pi r^2 h. \tag{7}$$

Because b << r, the items  $2(1+\alpha)\pi b^2 + \pi b^2 + (1+\alpha)\pi b^3$  in (7) can be omitted. (Is this a reasonable assumption? We will discuss this later.) Therefore, we have

$$SV(r,h) = \pi r b[2h + (1+\alpha)r], G(r,h) = \pi r^2 h - V.$$
(8)

We need to find the r and h so that the material volume SV(r, h) is minimized under the constraint G(r, h) = 0, that is, the mathematical model is

min  $SV(r,h) = \pi rb[2h + (1+\alpha)r]$ r > 0.h > 0

s.t. 
$$G(r,h) = \pi r^2 h - V = 0.$$
 (9)

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Step 5. Solving this mathematical model. We suggest two methods here. The first one is to solve *h* from G(r, h) = 0, that is,  $h = V / \pi r^2$ , and substitute it into SV(r, h), that is,  $SV(r) = b[2V / r + (1 + \alpha)\pi r^2]$ . Using calculus we can find the unique critical point of SV(r) as  $r_0 = \sqrt[3]{V/(1 + \alpha)\pi}$ , and it is easy to verify that  $S''(r_0) > 0$ . Therefore the material volume reaches its minimum at  $r_0$ . Furthermore, if we denote the corresponding height as  $h_0$ , then  $r_0 : h_0 = r_0 : (V / \pi r_0^2) = \pi r_0^3 / V = 1 : (1 + \alpha)$ .

Another method is to use the arithmetic mean and geometric mean inequality (elementary method, it also can be used for the high school students), that is,

$$\frac{1}{n}\sum_{i=1}^{n}a_{i} \geq \sqrt[n]{\prod_{i=1}^{n}a_{i}}, \ a_{i} > 0, \ i = 1, ..., n$$

Equality holds if and only if  $a_1 = a_2 = ... = a_n$ .

Specifically, taking n=3 and  $a_1 = a_2 = V/r$ ,  $a_3 = (1+\alpha)\pi r^2$ , we have  $SV(r) = b[2V/r + (1+\alpha)\pi r^2] \ge 3b \times \sqrt[3]{(V/r)^2(1+\alpha)\pi r^2} = 3b \times \sqrt[3]{(1+\alpha)\pi V^2}$ . The Equality holds if and only if  $V/r = (1+\alpha)\pi r^2$ , that is,  $r = \sqrt[3]{V/(1+\alpha)\pi}$ , so that we get the same result as in the first method.

Step 6. If  $\alpha = 3$  (If one measures the thickness of the top cover and the other part of the cover, it is indeed so), then the ratio *r*:*h*=1:4. The measurement roughly tells us that it is right!

The students will check if the model is right by using their measurement, and discuss if they need to modify the model through their imagination, while studying why omitting  $2(1+\alpha)\pi rb^2 + \pi hb^2 + (1+\alpha)\pi b^3$  is a reasonable assumption (this is not an easy problem, because if we do not omit these items, we have to find the zeros of a polynomial of degree three for finding critical points, namely,  $\frac{dSV(r)}{dr} = \frac{b}{r^3} \Big[ -2Vr + 2(1+\alpha)\pi r^4 + 2\pi(1+\alpha)br^3 - 2bV \Big] = 0$ . Students can use math

software for drawing pictures and obtaining numerical results to show it is indeed a reasonable assumption, especially students will get a strong impression about why they have to learn more mathematics in order to solve the problem and why math software is very helpful), and so on.

In fact, if we do not omit the higher order terms of *b*, we can also solve this problem analytically. Making use of  $h = V / \pi r^2$ , Equation (6) can be rewritten as

$$SV(r) = \pi b[(1+\alpha)(r+b)^2 + (2r+b)\frac{V}{\pi r^2}],$$
(10)

and therefore

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$$SV'(r) = 2\pi b(r+b)((1+\alpha) - \frac{V}{\pi r^3}), \ r_0 = \sqrt[3]{\frac{V}{(1+\alpha)\pi}},$$
(11)

$$SV''(r) = 2b(2\pi + \frac{(3b+2r)V}{r^4}) > 0, \ r > 0.$$
(12)

Finally,  $r_0 = \sqrt[3]{V/(1+\alpha)\pi}$  is the unique critical point and the result is still the same.

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### Discussion

The students will discuss if this model is right, or if they need to modify the model, the complexity of mathematical modelling etc. It is important that we leave room for students' imagination.

### **5. OTHER EXERCISES**

We design several exercises for students to practice the modelling process, especially, to get insight of the importance of mathematics and the necessity and convenience of using mathematical software, such as Mathematica, MATLAB, etc. In addition, we motivate students to think why we can believe the results from the use of the mathematical software to be correct. And we follow the principle: teach students in accordance with their aptitude. Two examples of the exercises are given in below.

1. If the thickness of the top cover of a beer can is four times thicker than the thickness of the other part of the can. The inside volume of the can is  $V = 500 \text{ cm}^3$ . What is the ratio of *r* and *h* of the can, which makes the volume of the can material minimum? (Go to the supermarket to see if there is a beer can which has the size from your modelling result.)

2. A Space Shuttle Water Container

Consider the space shuttle and an astronaut's water container that is stored within the shuttle's wall. The water container is formed as a sphere surmounted by a cone (like an ice cream cone), the base of which is equal to the radius of the sphere (see Figure 2). If the radius of the sphere (r) is restricted to exactly 6 ft and a surface area of 460 square feet is all that is allowed in the design, find the dimensions of the height ( $x_1$ ) of the spherical cap cutting by the cone and the height ( $x_2$ ) of the cone such that the volume of the container is maximized (See Giordano et al., 2003). This exercise is much harder but attracts the talented students.



Figure 2. The shape of the water container.

**Assessment.** (An example of the teamwork) Ask each team to write an essay on an optimization problem about a bowl shape (the frustum of a right cone) container (see Figure 3). Either one of the radiuses of the top base, lower base and the height

of the frustum of the corresponding right cone is known.

**Reading materials.** We also designed several reading materials on what is mathematical modelling and related topics such as isoperimetric problems in order to raise their interest in studying mathematics and to strengthen their mathematical knowledge. Reading materials also include some interesting stories, for example:

### The Legend of Princess Dido

According to the epic *Aeneid*, Dido (pronounced "Dee Dough") was a Phoenician princess from the city of Tyre (now part of Lebanon). Her treacherous brother, the king, murdered her husband, so she fled the city and sailed with some of her loyal subjects to Carthage, a city on the northern coast of Africa. She wished to purchase some land from the local ruler in order to begin a new life. However, he didn't like the idea of selling land to foreigners. In an attempt to be gracious and yet still spoil Princess Dido's request, the ruler said, "You may purchase as much land as you can enclose with the skin of an ox." Undaunted, Princess Dido and her subjects set about the task by slicing the ox skin into thin strips and then tying them together to form a long band of ox hide, and foiling the ruler's malicious plan (see Hildebrand & Anthony, 1985).

### REFERENCES

- Friedman, A., Glimm, J. and Lavery, J. (1992) The mathematical and computational sciences in emerging manufacturing technologies and management practices. SIAM Report on Issues in the Mathematical Sciences, SIAM, 62.
- Giordano, F.R., Weir, M.D. and Fox, W.P. (2003) A First Course in Mathematical Modelling, 3<sup>rd</sup> Edition. Brooks/Cole, 476-478.
- Hazewinkel, M. (ed) (1995) Encyclopedia of mathematics An updated and annotated translation of the Soviet mathematical encyclopedia. Dordrecht: Kluwer Academic Publishers, 3, 784-785.
- Hildebrand, S. and Anthony, T. (1985) *Mathematics and Optimal Form*. Scientific American Books.
- Stewart, J. (2003) Calculus, 5th Edition. Brooks/Cole, 332-333.
- Wan, F.Y.M. (1990) Mathematical models and their formulation. In C.E.Pearson, (ed) *Handbook of Applied Mathematics Selected Results and Methods*, 2<sup>nd</sup> Edition. Van Nostrand Reinhold, 1048.
- Ye, Q.X. (2003) Incorporating Ideas and Methods of Mathematical Modelling and Mathematical Experimentation into the Teaching of Calculus (in Chinese), *Engineering Mathematics*, 20, 8, 3-13.