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WOULD A FIRM BE MORE LIKELY TO USE BUY-ONE-GIVE-ONE AT A HIGHER PRODUCTION COST?

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Abstract. In this article, we consider the joint pricing and inventory policy of a profit-maximizing firm with a Buy-One-Give-One (BOGO) program, where the firm donates one unit of product for every unit sold. We find that the BOGO program doesn't necessarily require a higher inventory level. We also find that BOGO increases the price but does not increase the sales. Moreover, we find that BOGO is profitable for the firm only when the impact of the donation on consumer utility (the "warm glow" effect) is high enough. Contrary to intuition, the threshold of the "warm glow" effect does not necessarily increase with the firm's unit production cost. This means that in some cases, high-cost products may instead be more suitable for BOGO than low-cost products. We also consider three extension cases: ex-ante pricing, ex-post donating and competitive market.

1. **Introduction.** Corporate social responsibility (CSR) is a strategic concern for firms and has emerged as a global trend involving firms, states, international organizations and civil society organizations [29]. Governments provide subsidies to encourage firms to adopt socially responsible behavior [2]. At the same time, consumers are willing to pay more for socially responsible products and they expect socially responsible behavior from firms [17]. Governments, non-governmental organizations and consumers are putting pressure on firms to become more socially responsible as well [4]. Despite the growing importance of CSR, firms are often reluctant to invest in CSR programs due to the high financial burdens. Therefore, CSR is often considered to be applicable only to large firms [28]. In this context, a form of CSR called "Cause Marketing (CM)", which links charitable giving to product sales, has been proposed [31]. In a CM program, the firm donates a certain amount of cash for every product sold. CM offers a viable and sustainable way for small and medium-size firms to practice CSR. Indeed, the most visible element of CSR is often the charitable giving [12]. Firms have been donating more and more products to charities in recent years, and there is a rapidly growing share of non-cash donations [7]. Non-cash donation plays an increasingly important role in charitable giving and are widely used in CSR.

We focus on a special form of CM called "Buy-One-Give-One (BOGO)" in which, for every product sold, a socially responsible firm will donate a same product to

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people in need [34]. BOGO first became well known through TOMS Shoes. In 2006, TOMS introduced a "One for One" program, whereby for every pair of shoes sold, the firm donates a pair of new shoes to a child in need. Later in 2011, TOMS extended its "One for One" model to eyewear products [28]. While TOMS ended its BOGO program in 2019, the firm has donated over 60 million pairs of shoes and helped restore sight of about 400,000 people, and has grown from a start-up to a firm worth over a half-billion dollars [18]. With the success of TOMS, a growing number of firms introduce BOGO into their business models as a way to invest in social causes while improving product value, brand image and financial performance [10]. This Bar Saves Lives, who donates Plumpy'Nut to malnourished children around the world, has ran a BOGO program from 2013 till now. People Water has ran a BOGO program called "Drop for Drop", which donates clean drinking water to people and places in need, from 2006 till now. More examples are listed in Table 1.

TABLE 1. "Buy One Give One" example

Company	Product	Company	Product
361°	Shoes	Out of Print Clothing	Clothing, mugs, tote bags
Aurabeat	Air purifiers	People Water	Drinking water
Baby Teresa	Baby apparel	Project 7	Chewing gums
Better World Books	New and used books	Roma	Boots
Bixbee	Backpacks, lunch boxes, duffel bags	Roozt	Online marketplace
BoGo	Electric hand torches	Skylar	Perfume and candle
Bombas	Socks	Smile Squared	Toothbrushes
Cotopaxi	Outdoor apparel	Soapbox	Soap, body wash, hair care products
DOG for DOG	Dog food	The Little Bee Co.	Diapers
Everything Happy	blankets	This Bar Saves Lives	Health bars
Figs	Medical scrubs	Twice as Warm	Hats, scarves, gloves
Leesa	Mattress	Two Degree Food	Snack bars
Love Your Melon	Beanies	Warby Parker	Eyeglasses and sunglasses
One World Play Project	Soccer balls	WeWood	Wooden watches and glasses

With the popularity of BOGO around the world, a number of studies have explored how BOGO affects consumer response. According to Marquis and Park [23], BOGO links a firm's products/services to the social value of charitable giving, and enables consumers to become part of the social cause by purchasing. Hamby [15] shows that BOGO enhances consumer response through the perceived helpfulness of the donated entity. Dugan et al. [10] find that BOGO enhances purchase intentions for utilitarian products but undermines purchase intentions for hedonic products. Only few studies have discussed the firms' decision-making under BOGO promotion. Under deterministic conditions, Yen et al. [34] explore the price and quality decisions of a monopolistic firm with different BOGO formats, and compare them with the cross-subsidy model in terms of social welfare. Based on the random newsvendor model, Park et al. [28] examine the optimal inventory management of BOGO model and compare it to the cash donation model but without considering the impact of donations on consumer preference.

In this article, we construct a theoretical model consisting of a profit-maximizing firm and a random market where consumers are heterogeneous in their willingness-to-pay for the product. Based on previous research, BOGO can increase consumers' valuation of the product by giving them a "warm glow" from purchasing [2], and then the demand will increase. At the same time, BOGO increases the cost of the firm due to the donation for every product sold. Given the donation for every product sold and the fact that donations affect consumer utility and thus demand, it is natural to consider the impact of BOGO on the firm's inventory decision. On the other hand, given that donations increase the firm's cost while increasing consumer utility, it is also natural to consider the impact of BOGO on the firm's pricing decision. In addition, the pricing decision would affect the demand and thus the inventory decision.

With these in mind, we discuss the following questions. 1. How does BOGO affect the firm's pricing and inventory decisions? 2. When is BOGO beneficial to the firm? 3. Would the firm be more or less likely to use BOGO as the unit production cost increases?

We find that the BOGO program doesn't necessarily require a higher inventory level. This result is similar to Park et al. [28]. We also find that the BOGO program encourages the firm to charge a higher price. Moreover, we find that the BOGO program doesn't help to increase the sales. In fact, when the market size is small, the sales are the same whether there is a BOGO or not. When the market size is large, BOGO leads to lower sales.

By comparing the firm's profits with and without BOGO, we characterize the conditions under which the BOGO program is profitable for the firm. We find that BOGO is profitable only when the "warm glow" effect is higher than a threshold related to the firm's unit production cost. Furthermore, we find that this threshold is not necessarily always monotonically increasing in cost. Instead, for some demand distributions, the threshold decreases as the cost increases within a certain range. This means that in some cases, as the unit production cost increases, the firm is increasingly willing to use BOGO.

Finally, we consider three extensions. In the cases of ex-ante pricing and ex-post donating, the results are similar. In the case of a competitive market, we give a numerical example to to show the equilibrium BOGO strategies of the firms.

The rest of the article is organized as follows. In Section 2, we review the relevant literature. In Section 3, we describe the details of the model setting. In Sections 4 and 5, we give the pricing and inventory decisions of the firm with and without BOGO. In Section 6, we characterize the conditions under which it is profitable for the firm to use BOGO. In Section 7, we consider some extensions of the basic model. In Section 8, we conclude the article and suggest some directions for future research.

2. **Literature review.**

This article belongs to the literature of CSR research under BOGO. We also contribute to the operations management and marketing literature by investigating the firm's optimal pricing and inventory management problem.

2.1. **CSR.**

There is a lot of literature that examines the impact of CSR.

Some studies focus on the perspective of firms. Krishna and Rajan [19] suggest that CM products can not only improve the public perception of a firm but also lead to higher profit margins. Yang and Jiang [33] examine the impact of suppliers' CSR controversies on buyers' market value, as well as the moderating role of buyers' social capital. Chen and Delmas [5] propose a model to evaluate CSR performance from an efficiency perspective. Their studies are based on empirical analysis, and lack modeling of how CSR affects firms. Letizia and Hendrikse [21] study the relationship between supply chain structures and CSR adoption, but we focus on the impact of CSR on firm decision making. Albuquerque et al. [1] model CSR as an investment to increase product differentiation. They find that CSR decreases systematic risk and increases firm value. They consider a risk-aversion firm, while we focus on a risk-neutral case.

Some studies consider the consumer perspective. Sen and Bhattacharya [30] find that the effect of CSR on consumers' evaluation of a firm is contingent on their perceptions of the match between their own character and that of the firm. The effect of CSR may be positive or negative, depending on this congruence. Hildebrand et al. [16] examine consumer reactions to two basic contribution types, money versus in-kind, in the CSR domain of disaster relief. Mahmoudzadeh and Chaturvedia [24] study how consumer reactions are effective in improving a firm's uptake of responsibility practices in the sourcing domain. Park et al. [27] investigated whether and how CSR-linked sponsorship event type influences consumers' intention to purchase sponsors' products. The above studies are based on experiments, while our analysis is based on modeling. We consider the effect of BOGO on consumer utility as a starting point for our study. We assume that this effect is positive.

There is also literature concerned with social welfare. Arya and Mittendorf [2] study the consequences for supply chains when subsidies for CSR are offered. Yen et al. [34] explore different formats of BOGO and compare BOGO with cross-subsidy with regard to social welfare. They find that BOGO results in higher social welfare than cross-subsidy when either the social enterprise is highly socially responsible or the social gap between the rich and the poor is large.

2.2. **Pricing and inventory management.** This article is also closely related to the literature that examine pricing and inventory issues in the context of CSR.

A lot of studies focus on the firm's inventory management under CSR. Barcos et al. [3] study the impact of CSR on firms' inventory policy. They propose an inverted U-shaped relationship between firms' CSR and inventory levels. Elsayed [11] provides empirical evidence regarding the impact of inventory management on CSR. Park et al. [28] examine the optimal inventory management of the BOGO model under stochastic demand. They also compare it with the standard newsvendor model and the cash donation model. Cong et al. [8] establish a Stackelberg game model for a green supply chain consisting of a single supplier and a single manufacturer. They analyze the optimal results in terms of greenness level, production quantity, and social welfare and find that these tend to be higher when the party without CSR awareness is the leader. Ozbilge et al. [26] studies a socially responsible food-retailer's operational planning problem for a continuously deteriorating inventory over two periods with the consideration of donation and quality-sensitive customers. These studies do not consider the effect of price on firm demand. Considering the cost of donated products in the BOGO, the firm may need to raise price to cover the extra cost. Therefore, it is necessary to take pricing decisions into account.

There are also a number of studies that consider the pricing problem. Chen and Yang [6] and Gao [13] examine a firm's pricing decisions with CM and the impact on participating charity. They focus on cash donating, but we focus on product donating. Chu et al. [7] analyze the impact of the tax deduction for charitable donations on the optimal price and quantity decisions of a profit-driven firm. Their results show that the tax deduction, which is designed to encourage charitable donations, may lead to unexpected behavior by the firm. Lowrey et al. [22] examine the donation and pricing practices of competing firms. In their work, Chu et al. [7] and Lowrey et al. [22], the charitable donation is not related to sales, which is different from our work. Dai et al. [9] study the influence of reference price effects and CSR on the remanufacturing supply chain's pricing decision and social work donations. Li et al. [20] consider a supply chain consisting of a manufacturer performing CSR efforts and a retailer undertaking pre-sales service efforts. They develop a game-theoretic model to analyze the optimal channel delivery strategies and pricing and investment decisions of the manufacturer and the retailer.

2.3. **Comparison with literature.** There have been few modeling studies of BOGO. The most relevant for our research are the following five studies. Park et al. [28] examine the optimal inventory management of BOGO model. They don't consider the impact of price decision and BOGO on demand. Yen et al. [34] study the price and quality decisions of a monopolistic firm with different BOGO formats. Their study is based on a deterministic model that does not take uncertain demand into account. Chu et al. [7] analyze the impact of the tax deduction for charitable donations on the optimal price and quantity decisions. In their work, the charitable donation is not related to sales, which is different from BOGO. Chen and Yang [6] and Gao [13] focus on CM, in which the firm donates cash for every product sold. We compare this paper with the most relevant literature in Table 2.

TABLE 2. Comparison with literature

Paper	CSR type	Consumer heterogeneity	Pricing decision	Demand uncertainty	Inventory decision
Park et al. [28]	BOGO	×	×	✓	✓
Yen et al. [34]	BOGO	✓	✓	×	×
Chu et al. [7]	Charitable donation not related to sales	×	✓	✓	✓
Chen and Yang [6]	CM	×	✓	×	×
Gao [13]	CM	✓	✓	×	×
This paper	BOGO	✓	✓	✓	✓

A gap in the literature is that no BOGO modeling study considers both inventory and pricing decisions. In fact, BOGO affects both the pricing and inventory decisions: the firm needs to raise price to cover the cost of donated product and prepare additional inventory for donation. At the same time, the interplay between pricing and inventory decisions needs to be considered.

In this paper, we construct a stochastic demand influenced by price and BOGO strategy, starting from the utility of heterogeneous consumers. We include both inventory and pricing decisions in the model and investigate the effectiveness of BOGO. From the perspective of modeling study, we fill the gap in the existing literature.

3. **Model setup.** Consider a firm that produces and sells a single type of product to a mass of consumers with an uncertain market size.

The firm decides whether to offer a BOGO program and makes inventory and pricing decisions to maximize its expected profit.

The utility-maximizing consumers decide whether to buy.

3.1. **Firm segment.** We give some assumptions and notes about the firm.

Assumption 3.1. The firm has no fixed cost, and the unit production cost is c, where $c \in [0, 1]$.

Assumption 3.2. First of all, the firm has to decide whether to offer a BOGO program.

As a way to achieve social responsibility and enhance brand image, BOGO is usually a long-term strategy for the firm rather than a short-term promotional tool. For examples, from 2006 to 2019, TOMS has used a BOGO program called "One for One", which donates shoes to people in need, for 13 years. Since 2013 till now, This Bar Saves Lives has used a BOGO program, which donates Plumpy'Nut to malnourished children around the world, for 11 years. Since 2006 till now, People Water has used a BOGO program called "Drop for Drop", which donates clean drinking water to people and places in need, for 18 years.

Assumption 3.3. Secondly, the firm decides the inventory level Q before the market size S is observed. The market size S is a random variable that represents the total number of potential consumers entering the market.

Because of the long production lead time, the firm has to decides the inventory level Q in advance. At this point, the market size S is a random variable on $[A, B]$ with a cumulative distribution function $F(\cdot)$ and a probability density function $f(\cdot)$. We also assume that $0 \leq A < B \leq +\infty$, $F(s) = 0$ for $s \in [0, A]$, $F(s) = 1$ for $s \in [B, +\infty)$ and $0 < F(s) < 1$ for $s \in (A, B)$.

Assumption 3.4. Thirdly, the firm decides the selling price p after the actual market size s is observed.

Similar to Guo et al. [14], we use a price postponement assumption. The firm can postpone the pricing decision until production is complete and the selling season begins. During the production lead time, the firm can collect information to obtain the actual market size s.

3.2. **Consumer segment.** We give some assumptions and notes about the consumers.

Assumption 3.5. Each consumer purchases one or zero units of the product.

Assumption 3.6. Consumers are heterogeneous in their willingness-to-pay for the product. We denote it as θ, which is uniformly distributed over $[0, 1]$, i.e., $\theta \sim U[0, 1]$.

This is a simple and common assumption in the literature to characterize consumer utility, such as Lowrey et al. [22], Naeeni et al. [25] and Zhang et al. [35].

Assumption 3.7. A θ-type consumer will obtain an extra utility $\alpha\theta$ from the donated product if the firm offers a BOGO program.

This is because of the "warm glow" effect mentioned by Arya and Mittendorf [2], and α is a parameter that measures this effect.

According to these assumptions, the purchasing utility of a θ-type consumer can be given as

$$u(p, \theta) = \begin{cases} \theta - p, & if \quad \text{No BOGO}, \\ \\ \theta - p + \alpha\theta, & if \quad \text{With BOGO}. \end{cases} \tag{1}$$

Each consumer is rational and purchases one unit product only when the utility is nonnegative ($u \geq 0$). According to Equation (1), the total consumer demand is given as

$$D(s,p) = s \cdot Prob_\theta \{u(p,\theta) \geq 0\}$$
$$= \begin{cases} s\,(1-p)\,, & if \quad \text{No BOGO}, \quad 0 \leq p \leq 1, \\ \\ s\left(1 - \dfrac{p}{1+\alpha}\right), & if \quad \text{With BOGO}, \quad 0 \leq p \leq 1+\alpha. \end{cases} \tag{2}$$

3.3. **Summary of events and notations.** We summarize the sequence of events in the following two stages.

Stage 1: the market size S is a random variable.

(1) The firm decides whether to offer a BOGO program.

(2) The firm decides the inventory quantity Q.

Stage 2: the market size s is observed.

(3) The firm decides the selling price p.

(4) Consumers decide whether to purchase based on their utilities.

The main notations and their descriptions are given in Table 3.

Table 3. Summary of notations

Notation	Description
c	Firm's unit production cost, $c \in [0,1]$
θ	Consumer's willingness-to-pay for the product, $\theta \sim U[0,1]$
α	Consumer's warm glow effect parameter for donated product, $\alpha \in [0,1]$
S	Random market size, a random variable on $[A,B]$
$F(\cdot)$	Cumulative distribution function of S
$f(\cdot)$	Probability density function of S
s	Realized value of S
Q	Firm's inventory decision
Q_i^*	Firm's optimal inventory decision, $i = W$ (if With BOGO) and $i = N$ (if No BOGO)
p	Firm's pricing decision
$p_i(s,Q)$	Firm's optimal pricing decision under market size s and inventory level Q, $i \in \{W,N\}$
$p_i^*(s)$	Firm's optimal pricing decision under market size s and inventory level Q_i^*, $i \in \{W,N\}$
$D_i(s,p)$	Firm's total demand under market size s and price p, $i \in \{W,N\}$
$d_i(s,p,Q)$	Firm's actual sales under market size s, inventory level Q and price p, $i \in \{W,N\}$
$\hat{d}_i(s,Q)$	Firm's actual sales under market size s, inventory level Q and price $p_i(s,Q)$, $i \in \{W,N\}$
$d_i^*(s)$	Firm's actual sales under optimal inventory and pricing decisions, $i \in \{W,N\}$

4. **Pricing decision.** In this section, we discuss the pricing problem under an observed market size s and a given inventory level Q. We give the optimal pricing decisions for the scenarios with and without BOGO program respectively.

4.1. **No BOGO.** According to Equation (2), the total demand is

$$D_N(s,p) = s\,(1-p)\,, \quad if \quad 0 \leq p \leq 1, \tag{3}$$

when there is no BOGO program. The actual sales volume, denoted by d_N, cannot exceed the inventory level Q, so that we have

$$d_N(s,p,Q) = \min\{D_N(s,p), Q\} = \begin{cases} Q, & if \quad 0 \leq p \leq 1 - \dfrac{Q}{s}, \\ \\ s\,(1-p)\,, & if \quad 1 - \dfrac{Q}{s} \leq p \leq 1. \end{cases} \tag{4}$$

At stage 2, the inventory level is given, so that the cost of inventory is sunk. Then the firm's goal at this stage is to maximize the sales revenue $p \cdot d_N$. The optimal retail price for the scenario without BOGO program is given as

$$p_N(s,Q) = \arg\max_{0\le p\le 1} p \cdot d_N(s,p,Q) = \begin{cases} \frac{1}{2}, & if \quad A \le s \le 2Q, \\ 1 - \frac{Q}{s}, & if \quad 2Q \le s \le B. \end{cases} \tag{5}$$

Bring it into Equation (4), the actual sales volume for the scenario without BOGO program at the optimal pricing policy is given as

$$\hat{d}_N(s,Q) = d_N\left(s, p_N(s,Q), Q\right) = \begin{cases} \frac{s}{2}, & if \quad A \le s \le 2Q, \\ Q, & if \quad 2Q \le s \le B. \end{cases} \tag{6}$$

4.2. **With BOGO.** According to Equation (2), the total demand is

$$D_W(s,p) = s\left(1 - \frac{p}{1+\alpha}\right), \quad if \quad 0 \le p \le 1+\alpha, \tag{7}$$

when the firm offers a BOGO program. Because of the BOGO program, the firm should donate one unit from inventory for every consumer purchasing. The actual sales volume, denoted by d_W, cannot exceed half of the inventory level, so that we have

$$\begin{aligned} d_W(s,p,Q) &= \min\left\{D_W(s,p), \frac{Q}{2}\right\} \\ &= \begin{cases} \frac{Q}{2}, & if \quad 0 \le p \le \left(1 - \frac{Q}{2s}\right)(1+\alpha), \\ s\left(1 - \frac{p}{1+\alpha}\right), & if \quad \left(1 - \frac{Q}{2s}\right)(1+\alpha) \le p \le 1+\alpha. \end{cases} \end{aligned} \tag{8}$$

Same as the scenario without BOGO program, the firm's goal is to maximize the sales revenue. The optimal retail price for the scenario with BOGO program is given as

$$\begin{aligned} p_W(s,Q) &= \arg\max_{0\le p\le 1+\alpha} p \cdot d_W(s,p,Q) \\ &= \begin{cases} \frac{1+\alpha}{2}, & if \quad A \le s \le Q, \\ \left(1 - \frac{Q}{2s}\right)(1+\alpha), & if \quad Q \le s \le B. \end{cases} \end{aligned} \tag{9}$$

Bring it into Equation (8), the actual sales volume for the scenario with BOGO promotion at the optimal pricing policy is given as

$$\hat{d}_W(s,Q) = d_W\left(s, p_W(s,Q), Q\right) = \begin{cases} \dfrac{s}{2}, & if \quad A \le s \le Q, \\ \\ \dfrac{Q}{2}, & if \quad Q \le s \le B. \end{cases} \tag{10}$$

5. **Inventory decision.** In this section, we discuss the inventory problem under a random market size S. We also give the optimal inventory decisions for the scenarios with and without BOGO program respectively.

5.1. **No BOGO.** According to Equations (5) and (6), the firm's profit function with no BOGO promotion, denoted by π_N, is given as

$$\begin{aligned} \pi_N(s,Q) &= p_N(s,Q)\hat{d}_N(s,Q) - cQ \\ &= \begin{cases} \dfrac{1}{2}\cdot\dfrac{s}{2} - cQ, & if \quad A \le s \le 2Q, \\ \\ \left(1-\dfrac{Q}{s}\right)\cdot Q - cQ, & if \quad 2Q \le s \le B. \end{cases} \end{aligned} \tag{11}$$

Based on this, the expected profit of the firm, $\mathbb{E}_S\left[\pi_N(S,Q)\right]$, is given as

$$\mathbb{E}_S\left[\pi_N(S,Q)\right] = \int_0^{2Q} \frac{1}{2}\cdot\frac{s}{2} f(s)\mathrm{d}s + \int_{2Q}^{+\infty} \left(1-\frac{Q}{s}\right)\cdot Q f(s)\mathrm{d}s - cQ. \tag{12}$$

The firm decides the inventory level Q to maximize the expected profit. Let Q_N^* be the optimal inventory quantity for the scenario without BOGO program, i.e., the value of Q maximizing $\mathbb{E}_S[\pi_N(S,Q)]$.

$$Q_N^* = \arg\max_{Q\ge 0} \mathbb{E}_S[\pi_N(S,Q)].$$

For ease of presentation, we define

$$T(x) = \int_x^B \left(1-\frac{x}{s}\right) f(s)\mathrm{d}s, \quad x \in [0,B]. \tag{13}$$

Proposition 5.1. *With no BOGO promotion, we have the following results.*
(a) *The expected profit $\mathbb{E}_S[\pi_N(S,Q)]$ is concave in Q.*
(b) *The optimal inventory quantity, denoted by Q_N^*, is given by a solution of the first-order condition and can be written as*

$$Q_N^* = \frac{1}{2}T^{-1}(c). \tag{14}$$

(c) *Q_N^* is decreasing in c.*

Proof of Proposition 5.1. It is in Appendix A.1. □

With the optimal inventory level $Q_N^* = \frac{1}{2}T^{-1}(c)$, the optimal price under market size s is given as

$$p_N^*(s) = p_N(s, Q_N^*) = \begin{cases} \frac{1}{2}, & if \quad A \le s \le T^{-1}(c), \\ \\ 1 - \frac{T^{-1}(c)}{2s}, & if \quad T^{-1}(c) \le s \le B, \end{cases} \tag{15}$$

and the actual sales volume is given as

$$d_N^*(s) = \hat{d}_N(s, Q_N^*) = \begin{cases} \frac{s}{2}, & if \quad A \le s \le T^{-1}(c), \\ \\ \frac{1}{2}T^{-1}(c), & if \quad T^{-1}(c) \le s \le B. \end{cases} \tag{16}$$

5.2. **With BOGO.** According to Equations (9) and (10), the firm's profit function with BOGO promotion, denoted by π_W, is given as

$$\begin{aligned} \pi_W(s, Q) &= p_W(s, Q)\hat{d}_W(s, Q) - cQ \\ &= \begin{cases} \frac{1+\alpha}{2} \cdot \frac{s}{2} - cQ, & if \quad A \le s \le Q, \\ \\ \left(1 - \frac{Q}{2s}\right)(1+\alpha) \cdot \frac{Q}{2} - cQ, & if \quad Q \le s \le B. \end{cases} \end{aligned} \tag{17}$$

The expected profit of the firm, $\mathbb{E}_S[\pi_W(S, Q)]$, is given as

$$\mathbb{E}_S[\pi_W(S, Q)] = \int_0^Q \frac{1+\alpha}{2} \cdot \frac{s}{2} f(s) \mathrm{d}s + \int_Q^{+\infty} \left(1 - \frac{Q}{2s}\right)(1+\alpha) \cdot \frac{Q}{2} f(s) \mathrm{d}s - cQ. \tag{18}$$

Let Q_W^* be the optimal inventory quantity for the scenario with BOGO program, i.e., the value of Q maximizing $\mathbb{E}_S[\pi_W(S, Q)]$.

$$Q_W^* = \arg\max_{Q \ge 0} \mathbb{E}_S[\pi_W(S, Q)].$$

Similar to Proposition 5.1, we have results as Proposition 5.2.

Proposition 5.2. *When the firm offers BOGO, we have the following results.*
(a) *The expected profit $\mathbb{E}_S[\pi_W(S, Q)]$ is concave in inventory quantity Q.*
(b) *The optimal inventory quantity, denoted by Q_W^*, is given by a solution of the first-order condition and can be written as*

$$Q_W^* = \begin{cases} T^{-1}\left(\frac{2c}{1+\alpha}\right), & if \quad 0 \le c \le \frac{1+\alpha}{2}, \\ \\ 0, & if \quad \frac{1+\alpha}{2} < c \le 1. \end{cases} \tag{19}$$

(c) *Q_W^* is decreasing in c and increasing in α.*

Proof of Proposition 5.2. It is in Appendix A.2. □

When $0 \le c \le \frac{1+\alpha}{2}$, with optimal inventory level $Q_W^* = T^{-1}\left(\frac{2c}{1+\alpha}\right)$, the optimal pricing policy is given as

$$p_W^*(s) = p_W(s, Q_W^*)$$
$$= \begin{cases} \dfrac{1+\alpha}{2}, & if \quad A \le s \le T^{-1}\left(\dfrac{2c}{1+\alpha}\right), \\ \left[1 - \dfrac{T^{-1}\left(\frac{2c}{1+\alpha}\right)}{2s}\right](1+\alpha), & if \quad T^{-1}\left(\dfrac{2c}{1+\alpha}\right) \le s \le B, \end{cases} \tag{20}$$

and the actual sales volume is given as

$$d_W^*(s) = \hat{d}_W(s, Q_W^*) = \begin{cases} \dfrac{s}{2}, & if \quad A \le s \le T^{-1}\left(\dfrac{2c}{1+\alpha}\right), \\ \dfrac{1}{2}T^{-1}\left(\dfrac{2c}{1+\alpha}\right), & if \quad T^{-1}\left(\dfrac{2c}{1+\alpha}\right) \le s \le B. \end{cases} \tag{21}$$

5.3. **Comparison.** Comparing Q_N^* with Q_W^*, p_N^* with p_W^*, and d_N^* with d_W^*, we get the following corollaries.

Corollary 5.3. *If* $T^{-1}(2c) \ge \frac{1}{2}T^{-1}(c)$, *we have* $Q_W^* \ge Q_N^*$ *for any* $\alpha \in [0,1]$*; otherwise, if* $T^{-1}(2c) < \frac{1}{2}T^{-1}(c)$, *there exists a threshold value* $\bar{\alpha}$ *such that* $Q_W^* \ge Q_N^*$ *only when* $\alpha \ge \bar{\alpha}$.

Corollary 5.3 is true, since Q_W^* is increasing in α. Corollary 5.3 shows that the BOGO program doesn't necessarily require a higher inventory level.

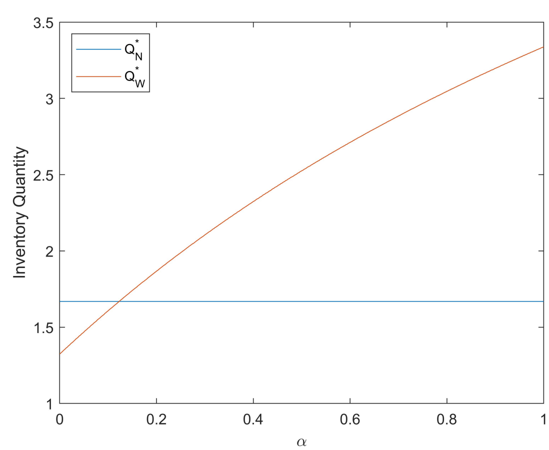

FIGURE 1. The inventory decisions with and without BOGO

We give an example where $S \sim U[0,10]$ and $c = 0.3$. As shown in Figure 1, when α is low, the donations do not increase consumers' willingness-to-pay enough. As a result, the demand could be severely affected by price increasing to cover the additional cost of BOGO, so that the firm with BOGO would instead produce less inventory than that without BOGO. Only when α is high enough, BOGO-induced increase in consumers' willingness-to-pay can cover this negative effect, would the firm with BOGO produce more.

Corollary 5.4. $p_W^* \geq (1+\alpha)p_N^*$.

Corollary 5.5. $d_W^* \leq d_N^*$.

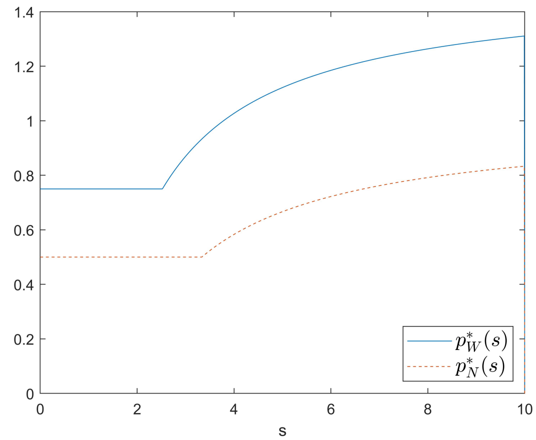

FIGURE 2. The pricing decisions with and without BOGO

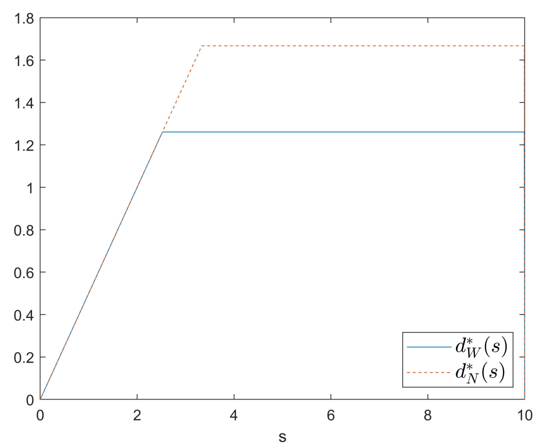

FIGURE 3. The actual sales with and without BOGO

We give an example where $S \sim U[0,10]$, $c = 0.3$ and $\alpha = 0.5$. As shown in Figure 2, with a BOGO program, the firm would price higher because of consumers' raising willingness-to-pay. However, as shown in Figure 3, the BOGO program doesn't help to increase sales. It is because of $Q_W^* \leq 2Q_N^*$, and so that the maximum saleable inventory with BOGO is less than that without BOGO.

6. **When is BOGO profitable.** As we have known the optimal inventory decisions Q_N^* (without BOGO) and Q_W^* (with BOGO), we can get the firm's maximum expected profits with and without BOGO program respectively.

For ease of presentation, we define

$$R(x) = \int_0^x s f(s)\mathrm{d}s + x^2 \int_x^B \frac{1}{s} f(s)\mathrm{d}s, \tag{22}$$

and $T(x)$ is given in Equation (13).

Proposition 6.1. *Given a random market size S, we have the following results.*
(a) *When there is no BOGO program, the firm's maximum expected profit, denoted by Π_N, can be written as*

$$\Pi_N = \frac{1}{4} R\left[T^{-1}(c)\right]. \tag{23}$$

(b) *When there is a BOGO program, the firm's maximum expected profit, denoted by Π_W, can be written as*

$$\Pi_W = \begin{cases} \frac{1+\alpha}{4} R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right], & if \quad 0 \le c \le \frac{1+\alpha}{2}, \\ \\ 0, & if \quad \frac{1+\alpha}{2} < c \le 1. \end{cases} \tag{24}$$

(c) *The firm will offer a BOGO program if and only if when the maximum expected profit with BOGO program is higher than that without BOGO program, i.e. , $\Pi_W \ge \Pi_N$. It is equal to*

$$(1+\alpha) R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] \ge R\left[T^{-1}(c)\right], \quad 0 \le c \le \frac{1+\alpha}{2}, \tag{25}$$

where $R(x)$ and $T(x)$ is same as given in part (a).

Proof of Proposition 6.1. It is in Appendix B.1. □

Proposition 6.1 gives the specific expressions for the firm's maximum expected profits with and without BOGO program. By comparing the maximum expected profits Π_N and Π_W, we can outline the conditions under which offering a BOGO program would be profitable. We then carry out an analysis of Equation (25).

Proposition 6.2.
(a) *For any fixed $c \in [0,1]$, there exists a threshold value $\tilde{\alpha}(c)$ such that*

$$\begin{cases} \Pi_W \le \Pi_N, & if \quad \alpha < \tilde{\alpha}(c), \\ \\ \Pi_W = \Pi_N, & if \quad \alpha = \tilde{\alpha}(c), \\ \\ \Pi_W \ge \Pi_N, & if \quad \alpha > \tilde{\alpha}(c). \end{cases}$$

Furthermore, we have

$$\tilde{\alpha}(c) \ge 2c - 1.$$

(b) *For any fixed $\alpha \in [0,1]$, there exists $\underline{c}(\alpha)$ and $\overline{c}(\alpha)$ such that*

$$\begin{cases} \Pi_W \geq \Pi_N, & if \quad c \leq \underline{c}(\alpha), \\ \Pi_W \leq \Pi_N, & if \quad c \geq \overline{c}(\alpha). \end{cases}$$

Furthermore, we have

$$\frac{1+\alpha}{2} T\left[R^{-1}\left(\frac{\mathbb{E}(S)}{1+\alpha}\right)\right] \leq \underline{c}(\alpha) \leq \overline{c}(\alpha) \leq \frac{1+\alpha}{2}.$$

Proof of Proposition 6.2. It is in Appendix B.2. □

The part (a) of Proposition 6.2 shows that for a given product (with fixed unit production cost c), the maximum expected profit with BOGO, Π_W, is increasing with the level of consumers' warm glow effect α. As α increases, the total demand would also increase, and the firm could then stock more and charge a higher price to make more profit. Only when such warm glow effect is high enough, i.e., $\alpha \geq \tilde{\alpha}(c)$, offering a BOGO program is profitable for the firm.

The part (b) of Proposition 6.2 shows that for any fixed α, it is profitable to offer a BOGO program when the cost c is low; and it is unprofitable when c is high. However, $\underline{c}$ is not necessarily equal to $\overline{c}$. For $c \in [\underline{c}, \overline{c}]$, the results are complicated, depending on the distribution of the random market size S. For example, $\alpha = 0.5$, S follows a distribution over $[0, 1] \cup [9, 10]$ such that uniformly distributed on $[0, 1]$ with probability 0.7 and uniformly distributed on $[9, 10]$ with probability 0.3. Then, Π_N and Π_W are as shown in Figure 4. Besides, $\underline{c} = 0.122$ and $\overline{c} = 0.284$.

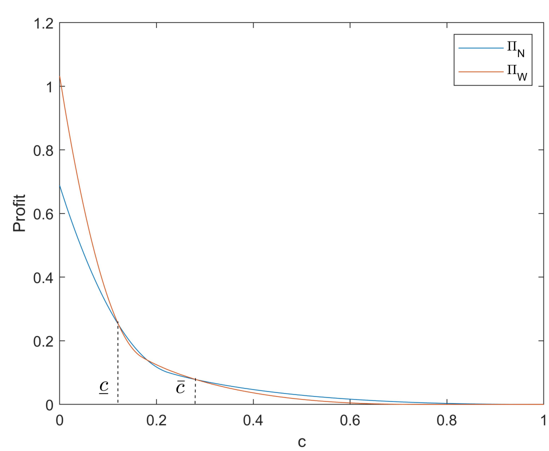

FIGURE 4. $\Pi_N(c)$ and $\Pi_W(c)$

Now we focus on $\tilde{\alpha}(c)$ and give two examples.

Example 1: S follows a uniform distribution over $[0, 10]$. This can be seen as the market of the products with relatively stable demand, such as daily necessities.

From Example 1, we find that for some distributions of S, as shown in Figure 5, $\tilde{\alpha}(c)$ is increasing with c. It means that the firm is less likely to use BOGO as production cost increases.

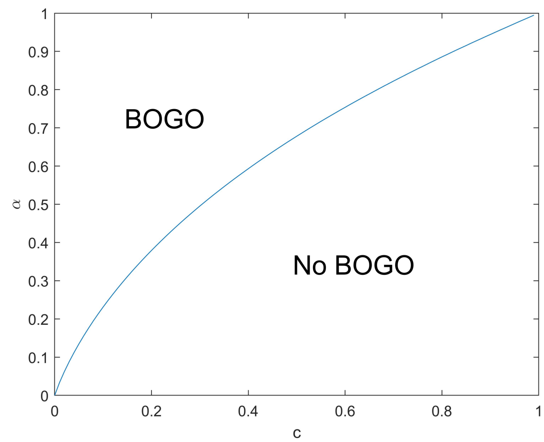

FIGURE 5. $\tilde{\alpha}(c)$ of Example 1

Example 2: S follows a distribution over $[0,1] \cup [9,10]$ such that uniformly distributed on $[0,1]$ with probability 0.7 and uniformly distributed on $[9,10]$ with probability 0.3. This can be seen as the market of the products with highly variable demand, such as down jackets, which are strongly affected by the weather.

From Example 2, we find that for some distributions of S, as shown in Figure 6, the relationship between $\tilde{\alpha}(c)$ and c is not simply incremental. Within a certain cost range, it is exactly the opposite. This means that in some cases, high-cost products may instead be more suitable for BOGO than low-cost ones.

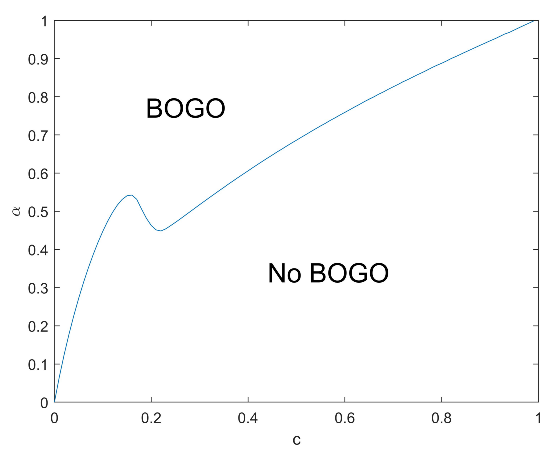

FIGURE 6. $\tilde{\alpha}(c)$ of Example 2

As shown in Example 2, there is a large gap between low demand and high demand. Because of the advance production for donating, BOGO program actually raises the inventory risk of the firm. When production cost c is low, the firm will focus on high demand first no matter BOGO is offered or not. At this point, as c

rises, BOGO is becoming less and less attractive for the firm, i.e., $\tilde{\alpha}(c)$ rises. When c increases further, the firm offering BOGO tends to give up on high-demand market due to inventory risk, while the firm without BOGO won't. As inventory pressure decreases, BOGO becomes profitable again, thus $\tilde{\alpha}(c)$ decreases. When c is high, the firm will give up high demand market no matter BOGO is offered or not. At this point, as c rises, BOGO is becoming less and less attractive for the firm, i.e., $\tilde{\alpha}(c)$ rises.

Proposition 6.3. *When $\tilde{\alpha}(c)$ is the threshold value given in Proposition 6.2 such that $\Pi_W = \Pi_N$ if $\alpha = \tilde{\alpha}(c)$, we have*

$$\tilde{\alpha}^{'}(c) \leq 0 \quad \Longleftrightarrow \quad 2c \cdot R\left[\frac{1}{2}T^{-1}(c)\right] \geq T\left[\frac{1}{2}T^{-1}(c)\right] \cdot R\left[T^{-1}(c)\right]. \tag{26}$$

Proof of Proposition 6.3. It is in Appendix B.3. □

It is hard to give an analytic expression for $\tilde{\alpha}(c)$, but we can sketch out some properties of $\tilde{\alpha}(c)$ through Equation (26).

7. **Extensions.** In this section, we consider some extensions of the basic model. First, we study a case in which the firm sets the price and the inventory level simultaneous before the market uncertainty is resolved. Second, we extend the model to a case in which the firm needs to prepare the inventory for selling in advance but could produce for donation later. Third, we extend the model to the competitive market.

7.1. **Ex-ante pricing.** In our basic model, we assume that the firm decides the inventory level in advance when the market size is unknown but decides the retail price after the market uncertainty is resolved. However, it is possible that the market information is hard to observe and the pricing decision may be a long-term policy for the firm in practice. To this end, we study a case with ex ante pricing, i.e., the firm makes the price decision before the market uncertainty is resolved.

7.1.1. *Exogenous price.* For a given price p, according to Equation (1), the firm's uncertain demands with BOGO and without BOGO, $\bar{D}_W(p)$ and $\bar{D}_N(p)$, are given as

$$\begin{cases} \bar{D}_N(p) = S\left(1-p\right), \\ \bar{D}_W(p) = S\left(1 - \dfrac{p}{1+\alpha}\right). \end{cases} \tag{27}$$

In this context, considering the inventory decisions of the firm, the model is similar to the newsvendor model. The firm's expected profits with BOGO and without BOGO, $\bar{\pi}_W(p,Q)$ and $\bar{\pi}_N(p,Q)$, are given as

$$\begin{cases} \bar{\pi}_N(p,Q) = p\mathbb{E}\left[\min\left\{\bar{D}_N, Q\right\}\right] - cQ, \\ \bar{\pi}_W(p,Q) = p\mathbb{E}\left[\min\left\{\bar{D}_W, \dfrac{Q}{2}\right\}\right] - cQ. \end{cases} \tag{28}$$

To maximize the expected profits, the optimal inventory level for fixed price p with BOGO and without BOGO, $\bar{Q}_W(p)$ and $\bar{Q}_N(p)$, are given as

$$\begin{cases} \bar{Q}_N(p) = (1-p)\, F^{-1}\left(\dfrac{p-c}{p}\right), \\ \\ \bar{Q}_W(p) = 2\left(1 - \dfrac{p}{1+\alpha}\right) F^{-1}\left(\dfrac{p-2c}{p}\right). \end{cases} \tag{29}$$

And then the maximal profits for fixed price p with BOGO and without BOGO, $\bar{\Pi}_W(p)$ and $\bar{\Pi}_N(p)$, are given as

$$\begin{cases} \bar{\Pi}_N(p) = p\,(1-p) \displaystyle\int_0^{F^{-1}\left(\frac{p-c}{p}\right)} sf(s)\mathrm{d}s, \\ \\ \bar{\Pi}_W(p) = p\left(1 - \dfrac{p}{1+\alpha}\right) \displaystyle\int_0^{F^{-1}\left(\frac{p-2c}{p}\right)} sf(s)\mathrm{d}s. \end{cases} \tag{30}$$

7.1.2. *Endogenous price.* When the price is endogenous for the firm, the optimal prices with BOGO and without BOGO, $\bar{p}_W$ and $\bar{p}_N$, are give as

$$\begin{cases} \bar{p}_N = \underset{c \le p \le 1}{\arg\max}\, \bar{\Pi}_N(p), \\ \\ \bar{p}_W = \underset{2c \le p \le 1+\alpha}{\arg\max}\, \bar{\Pi}_W(p). \end{cases} \tag{31}$$

And the maximal profits with BOGO and without BOGO, $\bar{\Pi}^*_W$ and $\bar{\Pi}^*_N$, are given as

$$\begin{cases} \bar{\Pi}^*_N = \bar{\Pi}_N\left(\bar{p}_N\right), \\ \\ \bar{\Pi}^*_W = \bar{\Pi}_W\left(\bar{p}_W\right). \end{cases} \tag{32}$$

Proposition 7.1. *There exists a threshold value $\bar{\alpha}(c)$ such that*

$$\begin{cases} \bar{\Pi}^*_W \le \bar{\Pi}^*_N, \quad if \quad \alpha < \bar{\alpha}(c), \\ \\ \bar{\Pi}^*_W = \bar{\Pi}^*_N, \quad if \quad \alpha = \bar{\alpha}(c), \\ \\ \bar{\Pi}^*_W \ge \bar{\Pi}^*_N, \quad if \quad \alpha > \bar{\alpha}(c). \end{cases} \tag{33}$$

Proof of Proposition 7.1. According to Equation (30), the $\bar{\Pi}_W(p)$ is increasing with α, so that the $\bar{\Pi}^*_W$ is also increasing with α. Therefore, the threshold value $\bar{\alpha}(c)$ exists. □

We give two examples about the $\bar{\alpha}(c)$ same as given in basic model.

Example 1: S follows a uniform distribution over $[0, 10]$.

As shown in Figure 7, the $\bar{\alpha}(c)$ is increasing with c, which means that the firm is less likely to use BOGO as production cost increases.

Example 2: S follows a distribution over $[0, 1] \cup [9, 10]$ such that uniformly distributed on $[0, 1]$ with probability 0.7 and uniformly distributed on $[9, 10]$ with probability 0.3.

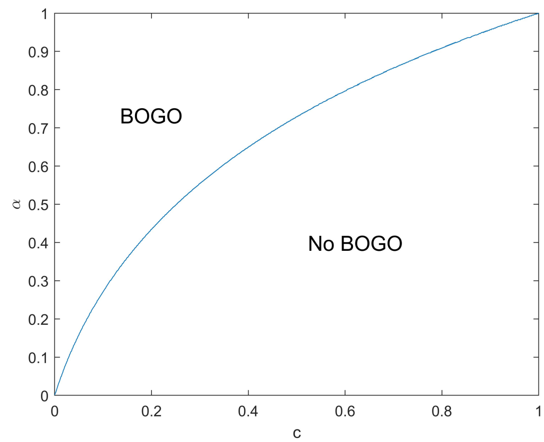

FIGURE 7. $\bar{\alpha}(c)$ of Example 1

As shown in Figure 8, the $\bar{\alpha}(c)$ is decreasing with c within a certain range of c. This means that sometimes high-cost products may instead be more suitable for BOGO than low-cost ones.

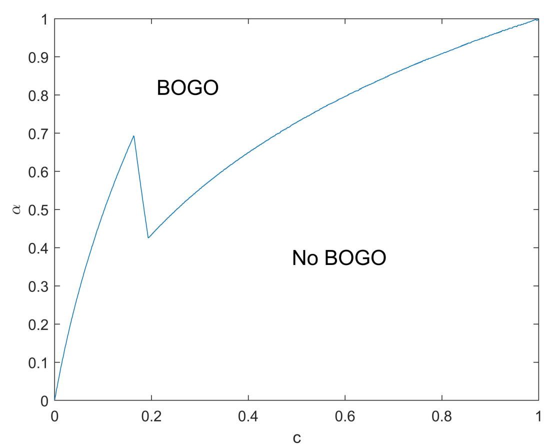

FIGURE 8. $\bar{\alpha}(c)$ of Example 2

From these two examples, we observe that this extended model has similar properties to the basic model.

7.2. **Ex-post donating.** We then consider a situation that the firm could produce the products for donation after the selling season. At that time, the sequence of events is given as follow. First, the firm decides whether to offer a BOGO program. Second, the firm sets the price and the inventory level before the selling season when the market size is uncertain. Third, the firm produces for donation if the firm offers a BOGO program and the leftover inventory cannot cover the donating commitment.

If the firm does not offer a BOGO program, the result is same as in Subsection 7.1. The optimal inventory level for fixed price p without BOGO can be written as

$$\check{Q}_N(p) = (1-p)\,F^{-1}\left(\frac{p-c}{p}\right), \tag{34}$$

and the maximal profits for fixed price p without BOGO can be written as

$$\check{\Pi}_N(p) = p\,(1-p)\int_0^{F^{-1}\left(\frac{p-c}{p}\right)} sf(s)\mathrm{d}s. \tag{35}$$

If the firm offers a BOGO program, for a given price p, the demand of the firm is

$$\check{D}_W(p) = S\left(1-\frac{p}{1+\alpha}\right). \tag{36}$$

Then the leftover inventory is

$$\check{L}_W(p,Q) = \max\left\{Q-\check{D}_W(p),0\right\}. \tag{37}$$

The donating quantity is the same as the sales quantity

$$Sales = \min\left\{Q,\check{D}_W(p)\right\}. \tag{38}$$

The number of additional products that the firm needs to produce is

$$Z = \max\left\{Sales-\check{L}_W(p,Q),0\right\}. \tag{39}$$

The expected profit of the firm is

$$\begin{aligned}\check{\pi}_W(p,Q) =&\mathbb{E}\left[p\cdot Sales - cZ - cQ\right]\\ =&p\left(1-\frac{p}{1+\alpha}\right)\int_0^{\frac{1+\alpha}{2(1+\alpha-p)}Q} sf(s)\mathrm{d}s + cQ\int_{\frac{1+\alpha}{2(1+\alpha-p)}Q}^{\frac{1+\alpha}{1+\alpha-p}Q} f(s)\mathrm{d}s\\ &+(p-2c)\left(1-\frac{p}{1+\alpha}\right)\int_{\frac{1+\alpha}{2(1+\alpha-p)}Q}^{\frac{1+\alpha}{1+\alpha-p}Q} sf(s)\mathrm{d}s\\ &+(p-c)Q\int_{\frac{1+\alpha}{1+\alpha-p}Q}^{+\infty} f(s)\mathrm{d}s - cQ.\end{aligned} \tag{40}$$

Then the optimal inventory level for fixed price p with BOGO can be written as

$$\check{Q}_W(p) = \left(1-\frac{p}{1+\alpha}\right)G^{-1}\left(\frac{p-2c}{p-c}\right), \tag{41}$$

where

$$G(x) = \frac{c}{p-c}F\left(\frac{x}{2}\right) + \frac{p-2c}{p-c}F(x). \tag{42}$$

Note that $G(x)$ can be seen as a cumulative distribution function for some other random variable. Let $g(x)$ be the corresponding probability density function,

$$g(x) = \frac{c}{2(p-c)}f\left(\frac{x}{2}\right) + \frac{p-2c}{p-c}f(x). \tag{43}$$

Then the maximal profit for fixed price p with BOGO can be written as

$$\check{\Pi}_W(p) = (p-c)\left(1-\frac{p}{1+\alpha}\right)\int_0^{G^{-1}\left(\frac{p-2c}{p-c}\right)} sg(s)\mathrm{d}s. \tag{44}$$

The optimal prices without BOGO and with BOGO, are given as

$$\begin{cases} \check{p}_N = \arg\max_{c \le p \le 1} \check{\Pi}_N(p), \\ \\ \check{p}_W = \arg\max_{2c \le p \le 1+\alpha} \check{\Pi}_W(p). \end{cases} \tag{45}$$

The maximal profits without BOGO and with BOGO, $\check{\Pi}_W^*$ and $\check{\Pi}_N^*$, are given as

$$\begin{cases} \check{\Pi}_N^* = \check{\Pi}_N\left(\check{p}_N\right), \\ \\ \check{\Pi}_W^* = \check{\Pi}_W\left(\check{p}_W\right). \end{cases} \tag{46}$$

Comparing $\check{\Pi}_N^*$ and $\check{\Pi}_W^*$, we have the following result.

Proposition 7.2. *There exists a threshold value $\check{\alpha}(c)$ such that*

$$\begin{cases} \check{\Pi}_W^* \le \check{\Pi}_N^*, & if \quad \alpha < \check{\alpha}(c), \\ \\ \check{\Pi}_W^* = \check{\Pi}_N^*, & if \quad \alpha = \check{\alpha}(c), \\ \\ \check{\Pi}_W^* \ge \check{\Pi}_N^*, & if \quad \alpha > \check{\alpha}(c). \end{cases} \tag{47}$$

Proof of Proposition 7.2. According to Equation (44), the $\check{\Pi}_W(p)$ is increasing with α, so that the $\check{\Pi}_W^*$ is also increasing with α. Therefore, the threshold value $\check{\alpha}(c)$ exists. □

We also give the same two examples for $\check{\alpha}(c)$.
Example 1: S follows a uniform distribution over $[0, 10]$.
In Example 1, as shown in Figure 9, the threshold $\check{\alpha}(c)$ is still increasing with c.

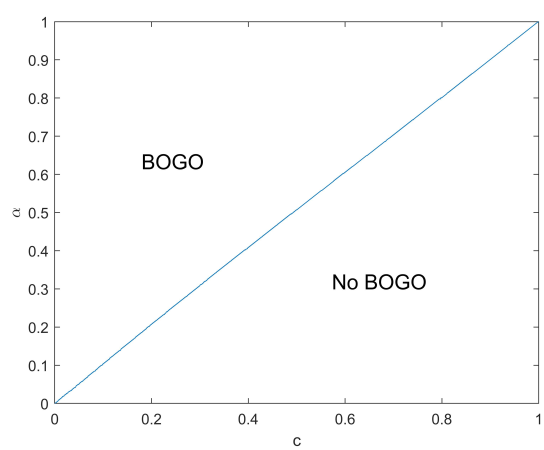

FIGURE 9. $\check{\alpha}(c)$ of Example 2

Example 2: S follows a distribution over $[0,1] \cup [9,10]$ such that uniformly distributed on $[0,1]$ with probability 0.7 and uniformly distributed on $[9,10]$ with probability 0.3.

In Example 2, as shown in Figure 10, the range in which $\check{\alpha}(c)$ decreases with c becomes smaller. This is due to the fact that the leftover inventory can be used for donating, BOGO program reduces the inventory risk for the firm. The firm can better balance high and low demand market conditions.

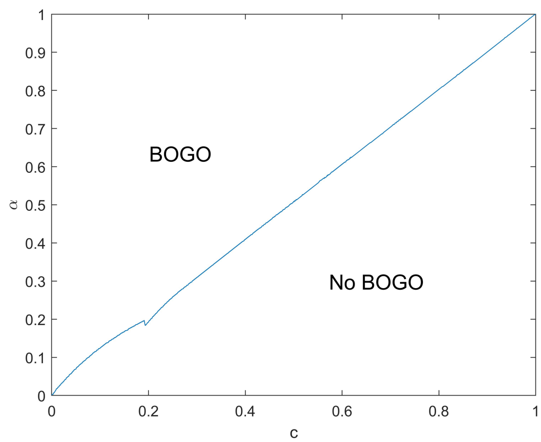

FIGURE 10. $\check{\alpha}(c)$ of Example 2

Corollary 7.3. $\check{\alpha}(c) \leq \bar{\alpha}(c)$.

Proof of Corollary 7.3. When $p \geq 2c$, $\check{\pi}_W(p,Q) \geq \bar{\pi}_W(p,Q)$, so that $\check{\Pi}_W(p) \geq \bar{\Pi}_W(p)$ and then $\check{\Pi}^*_W \geq \bar{\Pi}^*_W$. Besides, $\check{\Pi}^*_N = \bar{\Pi}^*_N$. Therefore, $\check{\alpha}(c) \leq \bar{\alpha}(c)$. ☐

This corollary implies that ex-post donating is helpful to increase profit of the firm. This makes the firm more likely to use BOGO.

7.3. **Competitive market.** In this subsection, we consider a competitive market consisting of utility-maximizing consumers and two profit-maximizing firms: Firm 1 and Firm 2. For Firm i, the quality of product is q_i and the unit production cost is c_i, where $i \in \{1,2\}$. We assume that $q_1 > q_2$ and $c_1 > c_2$.

The two firms play a three-stage game. In stage 1, the random market size S is unobserved, the firms decide their BOGO strategies, N (No BOGO) or W (With BOGO), at the same time. In stage 2, the random market size S is still unobserved, the firms decide their inventory quantities, Q_1 and Q_2, at the same time. In stage 3, the actual market size s is observed, the firms decide their prices, p_1 and p_2, at the same time.

The utility of a θ-type consumer purchasing from Firm i is

$$u_i(p_1,p_2,\theta) = \begin{cases} \theta q_i - p_i, & if \quad \text{No BOGO}, \\ \\ (1+\alpha)\theta q_i - p_i, & if \quad \text{With BOGO}. \end{cases} \tag{48}$$

So that when market size is s, the demand of Firm i is

$$D_i(p_1,p_2,s) = s\cdot Prob_\theta\left\{u_i(p_1,p_2,\theta)\geq \max\{0,u_j(p_1,p_2,\theta)\}, j\neq i\right\}, \tag{49}$$

and the actual sales is

$$d_i(p_1,p_2,s) = \begin{cases} \min\{D_i,Q_i\}, & if \quad \text{No BOGO},\\ \\ \min\{D_i,\dfrac{Q_i}{2}\}, & if \quad \text{With BOGO}.\end{cases} \tag{50}$$

In stage 3, for any fixed Q_1, Q_2 and s, the two firms solve the optimization problems

$$\begin{cases} \max\limits_{p_1} \quad p_1\cdot d_1(p_1,p_2,s),\\ \\ \max\limits_{p_2} \quad p_2\cdot d_2(p_1,p_2,s),\end{cases} \tag{51}$$

and then get the equilibrium prices $(p_1^*(s),p_2^*(s))$.

In stage 2, the two firms solve the optimization problems

$$\begin{cases} \max\limits_{Q_1}\Pi_1(Q_1,Q_2) = \displaystyle\int_0^{+\infty} p_1^*(s)d_1(p_1^*,p_2^*,s)f(s)\mathrm{d}s - c_1Q_1,\\ \\ \max\limits_{Q_2}\Pi_2(Q_1,Q_2) = \displaystyle\int_0^{+\infty} p_2^*(s)d_2(p_1^*,p_2^*,s)f(s)\mathrm{d}s - c_2Q_2,\end{cases} \tag{52}$$

and then get the equilibrium inventory levels (Q_1^*,Q_2^*) and equilibrium profits (Π_1^*,Π_2^*).

In stage 1, according to the BOGO strategies of the firms, we get four cases: (N, N), (W, N), (N, W) and (W, W). We also get the corresponding equilibrium profits: (Π_1^{NN},Π_2^{NN}), (Π_1^{WN},Π_2^{WN}), (Π_1^{NW},Π_2^{NW}) and (Π_1^{WW},Π_2^{WW}). By comparing the profits of the four cases, we can get the equilibrium BOGO strategies.

The analytical solution is not easy to obtained, we give an example to show how the BOGO strategies of firms change with α. We assume that S follows a uniform distribution over [0, 10], $q_1 = 3$, $c_1 = 0.6$, $q_2 = 1$ and $c_2 = 0.2$. We solve this example numerically and obtain the following results.

7.3.1. *Case (N,N).* In this case, both firms sell without BOGO program. By solving problem (51) in stage 3 and then problem (52) in stage 2, we get

$$\left(\Pi_1^{NN},\Pi_2^{NN}\right) = (1.54, 0.16).$$

7.3.2. *Case (W,N).* In this case, Firm 1 sells with BOGO program and Firm 2 sells without BOGO program. Then $\left(\Pi_1^{WN},\Pi_2^{WN}\right)$ is shown in Figure 11.

We can see that Π_1^{WN} increases with α and Π_2^{WN} decreases with α. This reflects the role of BOGO as a competitive tool that increases with α.

Besides, when α is low, $\Pi_1^{WN} < \Pi_1^{NN}$ and $\Pi_2^{WN} > \Pi_2^{NN}$. The extra utility of consumer getting from donation is low, so that Firm 1 cannot raise price high enough to cover the extra cost of donation. At that time, BOGO reduces the competitiveness of Firm 1. As α increases, (W, N) becomes a win-win strategy. The firms share the extra profit created by BOGO. When α is high, the competitiveness of Firm 1 is further enhanced, $\Pi_1^{WN} > \Pi_1^{NN}$ and $\Pi_2^{WN} < \Pi_2^{NN}$.

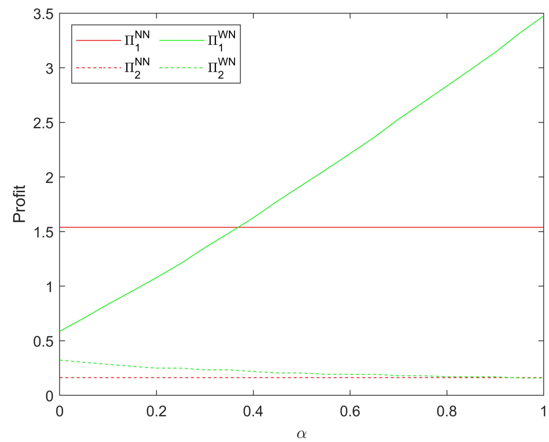

FIGURE 11. Case (W, N)

7.3.3. *Case (N, W).* In this case, Firm 1 sells without BOGO program and Firm 2 sells with BOGO program. Then $\left(\Pi_1^{NW}, \Pi_2^{NW}\right)$ is shown in Figure 12.

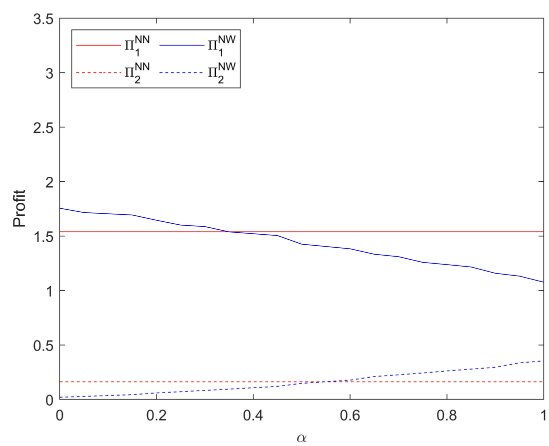

FIGURE 12. Case (N, W)

Different from case (W, N), when α is in the middle region, (N, W) becomes a lose-lose strategy. This is because BOGO enhances the weaker firm and brings the competitiveness of the two firms closer to each other, which in turn intensifies market competition.

7.3.4. *Case (W, W).* In this case, both firms sell with BOGO program. Then $\left(\Pi_1^{WW}, \Pi_2^{WW}\right)$ is shown in Figure 13.

When α is low, (W W) is a lose-lose strategy. When α is high, (W W) becomes a win-win strategy.

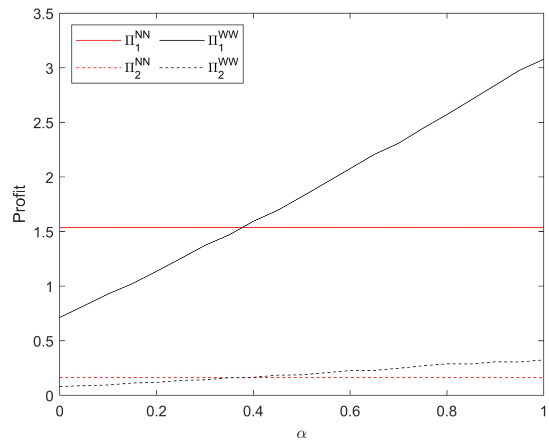

FIGURE 13. Case (W, W)

7.3.5. *Equilibrium BOGO strategy.* By comparing the four cases, we get the equilibrium BOGO strategy of the two firms as shown in Table 4.

TABLE 4. Equilibrium BOGO strategy

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Equilibrium in Stage 1	(N, N)	(N, N)	(N, N)	(N, N)	(W, N)	(W, N)	(W, N)	(W, W)	(W, W)	(W, W)	(W, W)

When α is low, neither firm offers a BOGO program. As α increases, when α is in the middle region, Firm 1 will offer a BOGO program but Firm 2 will not. This suggests that in competition, the dominant firm prefers to use BOGO. When α is high, the benefit of BOGO is high enough, so that both firms are willing to use BOGO.

8. **Conclusion.** As a potential and sustainable CSR practice for small and medium-size firms, BOGO is receiving more and more attention worldwide but has not been extensively discussed in the existing literature. It is natural to explore the pricing and inventory problem under BOGO, since the donated products inevitably affect the inventory decision and the cost, which in turn affect the pricing decision. We also compare the profits of the firm with and without BOGO, and give the conditions under which having BOGO is better.

In this article, we propose a stochastic single-product model in which consumer purchases are influenced by price and donation. We call the effect of donations on consumer utility as “warm glow” effect. Comparing the optimal pricing and inventory policies with and without BOGO, we find that BOGO doesn’t necessarily require a higher inventory level. It depends on the level of the “warm glow” effect. Only when the “warm glow” effect is high, does the BOGO lead to an inventory level higher than that without BOGO. In addition, BOGO leads to a higher price because of the higher consumer valuations of products. However, BOGO doesn’t help to increase the sales. Especially when the market size is large, BOGO instead reduces the sales.

Furthermore, we compare the profits of the firm with and without BOGO. We find that for any fixed unit production cost, the BOGO is profitable only when the "warm glow" effect is high enough. Surprisingly, we find that the threshold of "warm glow" effect does not necessarily increase with the production cost. Contrary to intuition, when the random market size follows some special distributions, the threshold instead decreases with cost within a certain range. This means that sometimes high-cost products may instead be more suitable for BOGO than low-cost products.

Finally, we explore three extensions of the basic model. In the ex-ante pricing case, the results are similar to the basic model. In the ex-post donating case, due to the fact that the leftover inventory can be used for donating, BOGO program reduces the inventory risk for the firm. So the threshold value of α reduces. In the competitive case, we give numerical example to to show how the equilibrium BOGO strategies change with α.

Our study has several limitations. First, as a potential emerging business model, the role of BOGO in competition is worth exploring. However, we provide only a numerical example in this article. Second, we examine the retailer's pricing and inventory decisions, but do not discuss the impact of BOGO on the supplier. The impact of BOGO on supply chain could also be a direction for future research. Third, our article only considers the demand uncertainty. In addition, the supply uncertainty is also an extendable direction. Fourth, it is interesting to consider if a consumer can be also a recipient. Finally, by adjusting the decision sequence, we can also consider a case where the firm could decide whether to use BOGO or not after observing the market situation.

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Appendix A. Proof of Propositions in Section 5.

A.1. Proof of Proposition 5.1.

Proof. According to Equation (12),

$$\mathbb{E}_S[\pi_N(S,Q)] = \int_0^{2Q} \frac{1}{2} \cdot \frac{s}{2} f(s)\mathrm{d}s + \int_{2Q}^{+\infty} \left(1 - \frac{Q}{s}\right) \cdot Q f(s)\mathrm{d}s - cQ.$$

(a) Differentiating $\mathbb{E}_S[\pi_N(S,Q)]$ with respect to Q, we have

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}_S[\pi_N(S,Q)]}{\mathrm{d}Q} &= \frac{2Q}{4} f(2Q) \cdot 2 - \left(1 - \frac{Q}{2Q}\right) Q f(2Q) \cdot 2 \\ &\quad + \int_{2Q}^{+\infty} \left(1 - \frac{2Q}{s}\right) f(s)\mathrm{d}s - c \\ &= \int_{2Q}^{+\infty} \left(1 - \frac{2Q}{s}\right) f(s)\mathrm{d}s - c.\end{aligned}$$

Let $T(x) = \int_x^B \left(1 - \frac{x}{s}\right) f(s)\mathrm{d}s$, then we have

$$\frac{\mathrm{d}\mathbb{E}_S[\pi_N(S,Q)]}{\mathrm{d}Q} = \begin{cases} T(2Q) - c, & if \quad 2Q \le B, \\ \\ -c, & if \quad 2Q > B. \end{cases} \tag{53}$$

Furthermore,

$$\begin{aligned} \frac{\mathrm{d}^2\mathbb{E}_S[\pi_N(S,Q)]}{\mathrm{d}Q^2} &= -\left(1 - \frac{2Q}{2Q}\right) f(2Q) \cdot 2 + \int_{2Q}^{+\infty} \left(-\frac{2}{s}\right) f(s)\mathrm{d}s \\ &= -2\int_{2Q}^{+\infty} \frac{1}{s} f(s)\mathrm{d}s \\ &\le 0. \end{aligned}$$

Thus, the expected profit $\mathbb{E}_S[\pi_N(S,Q)]$ is concave in Q.
(b) Note that $T(0) = \int_0^{+\infty} f(s)\mathrm{d}s = 1$, and $T(B) = 0$. Besides,

$$T^{'}(x) = -\int_x^B \frac{1}{s} f(s)\mathrm{d}s < 0, \quad \forall x < B,$$

so that $T(x)$ is monotonically decreasing in x. Combined with Equation (53), as $0 \le c \le 1$, we have

$$\left.\frac{\mathrm{d}\mathbb{E}_S[\pi_N(S,Q)]}{\mathrm{d}Q}\right|_{Q=0} = 1 - c \ge 0,$$

$$\left.\frac{\mathrm{d}\mathbb{E}_S[\pi_N(S,Q)]}{\mathrm{d}Q}\right|_{Q \ge \frac{B}{2}} = -c < 0.$$

Then, by the concavity of $\mathbb{E}_S[\pi_N(S,Q)]$, the optimal inventory level Q_N^* must satisfy the first-order condition by setting Equation (53) to zero. So that we have

$$Q_N^* = \frac{1}{2} T^{-1}(c).$$

(c) As $T(x)$ is decreasing in x, we have $Q_N^* = \frac{1}{2} T^{-1}(c)$ is decreasing in c.

□

A.2. **Proof of Proposition 5.2.**

Proof. It is similar to the Proof of Proposition 5.1.
According to Equation (18),

$$\mathbb{E}_S\left[\pi_W(S,Q)\right] = \int_0^Q \frac{1+\alpha}{2} \cdot \frac{s}{2} f(s)\mathrm{d}s + \int_Q^{+\infty} \left(1 - \frac{Q}{2s}\right)(1+\alpha) \cdot \frac{Q}{2} f(s)\mathrm{d}s - cQ.$$

(a) Differentiating $\mathbb{E}_S[\pi_W(S,Q)]$ with respect to Q, we have

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q} &= \frac{1+\alpha}{2}\cdot\frac{Q}{2}f(Q) - \left(1-\frac{Q}{2Q}\right)(1+\alpha)\cdot\frac{Q}{2}f(Q) \\ &\quad + \int_Q^{+\infty}\left(1-\frac{Q}{s}\right)\frac{1+\alpha}{2}f(s)\mathrm{d}s - c \\ &= \frac{1+\alpha}{2}\int_Q^{+\infty}\left(1-\frac{Q}{s}\right)f(s)\mathrm{d}s - c.\end{aligned}$$

It can be written as

$$\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q} = \begin{cases}\frac{1+\alpha}{2}T(Q) - c, & if \quad Q \le B, \\ \\ -c, & if \quad Q > B.\end{cases} \tag{54}$$

Furthermore,

$$\begin{aligned}\frac{\mathrm{d}^2\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q^2} &= \frac{1+\alpha}{2}\left[-\left(1-\frac{Q}{Q}\right)f(Q) + \int_Q^{+\infty}\left(-\frac{1}{s}\right)f(s)\mathrm{d}s\right] \\ &= -\frac{1+\alpha}{2}\int_Q^{+\infty}\frac{1}{s}f(s)\mathrm{d}s \\ &\le 0.\end{aligned}$$

Thus, the expected profit $\mathbb{E}_S[\pi_W(S,Q)]$ is concave in Q.
(b) Note that $T(0) = \int_0^{+\infty} f(s)\mathrm{d}s = 1$, $T(B) = 0$, and $T(x)$ is monotonically decreasing in x. Combined with Equation (54), if $0 \le c \le \frac{1+\alpha}{2}$, we have

$$\left.\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q}\right|_{Q=0} = \frac{1+\alpha}{2} - c \ge 0,$$

$$\left.\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q}\right|_{Q\ge B} = -c < 0.$$

Then, by the concavity of $\mathbb{E}_S[\pi_W(S,Q)]$, the optimal inventory level Q_W^* must satisfy the first-order condition by setting Equation (54) to zero. So that we have

$$Q_W^* = T^{-1}\left(\frac{2c}{1+\alpha}\right), \quad if \quad 0 \le c \le \frac{1+\alpha}{2}.$$

If $\frac{1+\alpha}{2} < c \le 1$, we have

$$\left.\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q}\right|_{Q=0} = \frac{1+\alpha}{2} - c < 0.$$

Note that $T(x)$ is decreasing in x, so that $\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q}$ is decreasing in Q. Then we have

$$\frac{\mathrm{d}\mathbb{E}_S[\pi_W(S,Q)]}{\mathrm{d}Q} \le \frac{1+\alpha}{2} - c < 0, \quad \forall Q \ge 0.$$

Therefore, the optimal inventory level

$$Q_W^* = 0, \quad if \quad \frac{1+\alpha}{2} < c \leq 1.$$

(c) As $T(x)$ is decreasing in x, we have $Q_W^* = T^{-1}\left(\frac{2c}{1+\alpha}\right)$ is decreasing in c and increasing in α.

□

Appendix B. Proof of Propositions in Section 6.

B.1. Proof of Proposition 6.1.

Proof. (a) Without BOGO program, the maximum expect profit is $\Pi_N = \mathbb{E}_S\left[\pi_N(S, Q_N^*)\right]$. According to Equation (12),

$$\mathbb{E}_S\left[\pi_N(S,Q)\right] = \int_0^{2Q} \frac{1}{2}\cdot\frac{s}{2}f(s)\mathrm{d}s + \int_{2Q}^{+\infty}\left(1-\frac{Q}{s}\right)\cdot Qf(s)\mathrm{d}s - cQ.$$

Then, according to Equation (14), we have

$$Q_N^* = \frac{1}{2}T^{-1}(c) \leq \frac{B}{2},$$

then we have

$$c = T(2Q_N^*) = \int_{2Q_N^*}^{B}\left(1-\frac{2Q_N^*}{s}\right)f(s)\mathrm{d}s,$$

and then

$$cQ_N^* = Q_N^*\int_{2Q_N^*}^{B}\left(1-\frac{2Q_N^*}{s}\right)f(s)\mathrm{d}s.$$

Bringing it into $\mathbb{E}_S\left[\pi_N(S,Q_N^*)\right]$, we get

$$\begin{aligned}
\Pi_N =& \mathbb{E}_S\left[\pi_N(S,Q_N^*)\right]\\
=&\frac{1}{4}\int_0^{2Q_N^*} sf(s)\mathrm{d}s + Q_N^*\int_{2Q_N^*}^{B}\left(1-\frac{Q_N^*}{s}\right)f(s)\mathrm{d}s - cQ_N^*\\
=&\frac{1}{4}\int_0^{2Q_N^*} sf(s)\mathrm{d}s + Q_N^*\int_{2Q_N^*}^{B}\left(1-\frac{Q_N^*}{s}\right)f(s)\mathrm{d}s\\
&- Q_N^*\int_{2Q_N^*}^{B}\left(1-\frac{2Q_N^*}{s}\right)f(s)\mathrm{d}s\\
=&\frac{1}{4}\int_0^{2Q_N^*} sf(s)\mathrm{d}s + Q_N^*\int_{2Q_N^*}^{B}\frac{Q_N^*}{s}f(s)\mathrm{d}s\\
=&\frac{1}{4}\left[\int_0^{2Q_N^*} sf(s)\mathrm{d}s + (2Q_N^*)^2\int_{2Q_N^*}^{B}\frac{1}{s}f(s)\mathrm{d}s\right].
\end{aligned}$$

So that

$$\Pi_N = \frac{1}{4}R(2Q_N^*) = \frac{1}{4}R\left[T^{-1}(c)\right],$$

where

$$R(x) = \int_0^x sf(s)\mathrm{d}s + x^2\int_x^B \frac{1}{s}f(s)\mathrm{d}s.$$

Furthermore,

$$R^{'}(x) = xf(x) + 2x\int_x^B \frac{1}{s}f(s)\mathrm{d}s - x^2\left[\frac{1}{x}f(x)\right] = 2x\int_x^B \frac{1}{s}f(s)\mathrm{d}s \geq 0,$$

so that $R(x)$ is increasing in x.
(b) With BOGO program, the maximum expect profit is $\Pi_W = \mathbb{E}_S\left[\pi_W(S, Q_W^*)\right]$. According to Equation (18),

$$\mathbb{E}_S[\pi_W(S,Q)] = \int_0^Q \frac{1+\alpha}{2}\cdot\frac{s}{2}f(s)\mathrm{d}s + \int_Q^{+\infty}\left(1-\frac{Q}{2s}\right)(1+\alpha)\cdot\frac{Q}{2}f(s)\mathrm{d}s - cQ,$$

Then, according to Equation (19), if $\frac{1+\alpha}{2} < c \leq 1$, we have $Q_W^* = 0$, then $\Pi_W = \mathbb{E}_S\left[\pi_W(S,0)\right] = 0$. If $0 \leq c \leq \frac{1+\alpha}{2}$, as $Q_W^* = T^{-1}\left(\frac{2c}{1+\alpha}\right) \leq B$, we have

$$\frac{2c}{1+\alpha} = T(Q_W^*) = \int_{Q_W^*}^B \left(1-\frac{Q_W^*}{s}\right)f(s)\mathrm{d}s,$$

and then we have

$$cQ_W^* = \frac{1+\alpha}{2}Q_W^*\int_{Q_W^*}^B \left(1-\frac{Q_W^*}{s}\right)f(s)\mathrm{d}s.$$

Bringing it into $\mathbb{E}_S\left[\pi_W(S,Q_W^*)\right]$, we have

$$\begin{aligned}
\Pi_W =&\mathbb{E}_S\left[\pi_W(S,Q_W^*)\right]\\
=&(1+\alpha)\frac{1}{4}\int_0^{Q_W^*} sf(s)\mathrm{d}s + (1+\alpha)\frac{Q_W^*}{2}\int_{Q_W^*}^B\left(1-\frac{Q_W^*}{2s}\right)f(s)\mathrm{d}s - cQ_W^*\\
=&(1+\alpha)\frac{1}{4}\int_0^{Q_W^*} sf(s)\mathrm{d}s + (1+\alpha)\frac{Q_W^*}{2}\int_{Q_W^*}^B\left(1-\frac{Q_W^*}{2s}\right)f(s)\mathrm{d}s\\
&-\frac{1+\alpha}{2}Q_W^*\int_{Q_W^*}^B\left(1-\frac{Q_W^*}{s}\right)f(s)\mathrm{d}s\\
=&(1+\alpha)\frac{1}{4}\int_0^{Q_W^*} sf(s)\mathrm{d}s + (1+\alpha)\frac{Q_W^*}{2}\int_{Q_W^*}^B \frac{Q_W^*}{2s}f(s)\mathrm{d}s\\
=&(1+\alpha)\cdot\frac{1}{4}\left[\int_0^{Q_W^*} sf(s)\mathrm{d}s + {Q_W^*}^2\int_{Q_W^*}^B \frac{1}{s}f(s)\mathrm{d}s\right],
\end{aligned}$$

then we get

$$\Pi_W = (1+\alpha)\cdot\frac{1}{4}R(Q_W^*) = (1+\alpha)\cdot\frac{1}{4}R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right].$$

(c) The firm will offer a BOGO program if and only if $\Pi_W \geq \Pi_N$.
If $\frac{1+\alpha}{2} \leq c \leq 1$, $\Pi_N = \frac{1}{4}R\left[T^{-1}(c)\right]$, $\Pi_W = 0$, $\Pi_W \leq \Pi_N$.
If $0 \leq c \leq \frac{1+\alpha}{2}$, $\Pi_N = \frac{1}{4}R\left[T^{-1}(c)\right]$, $\Pi_W = (1+\alpha)\cdot\frac{1}{4}R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right]$, so that $\Pi_W \geq \Pi_N$ if and only if $(1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] \geq R\left[T^{-1}(c)\right]$.

□

B.2. **Proof of Proposition 6.2.**

Proof. According to Equation (25),

$$\Pi_W \geq \Pi_N \iff (1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] \geq R\left[T^{-1}(c)\right], \quad 0 \leq c \leq \frac{1+\alpha}{2}.$$

(a) For any fixed $c \in [0,1]$, $R\left[T^{-1}(c)\right]$ is fixed. Let

$$g(\alpha) = (1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right], \quad \alpha \in [\max\{2c-1,0\},1],$$

then

$$\Pi_W \geq \Pi_N \iff g(\alpha) \geq R\left[T^{-1}(c)\right].$$

If $c \geq \frac{1}{2}$, we have

$$g(2c-1) = 2cR\left[T^{-1}(1)\right] = 2cR(0) = 0 \leq R\left[T^{-1}(c)\right].$$

If $c \leq \frac{1}{2}$, as $T(x)$ is a decreasing function, we have $T^{-1}(2c) \leq T^{-1}(c)$. As $R(x)$ is an increasing function, we have

$$g(0) = R\left[T^{-1}(2c)\right] \leq R\left[T^{-1}(c)\right].$$

Thus, we have

$$g(\max\{2c-1,0\}) \leq R\left[T^{-1}(c)\right] \leq 2R\left[T^{-1}(c)\right] = g(1).$$

Furthermore,

$$\begin{aligned}
g^{'}(\alpha) &= R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] \\
&\quad + (1+\alpha)R^{'}\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right]\left(T^{-1}\right)^{'}\left(\frac{2c}{1+\alpha}\right)\left[-\frac{2c}{(1+\alpha)^2}\right] \\
&= R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] - \frac{2c}{1+\alpha}\cdot\frac{R^{'}\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right]}{T^{'}\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right]} \\
&= R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] - \frac{2c}{1+\alpha}\cdot\frac{2T^{-1}\left(\frac{2c}{1+\alpha}\right)\int_{T^{-1}\left(\frac{2c}{1+\alpha}\right)}^{B}\frac{1}{s}f(s)\mathrm{d}s}{-\int_{T^{-1}\left(\frac{2c}{1+\alpha}\right)}^{B}\frac{1}{s}f(s)\mathrm{d}s} \\
&= R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] + \frac{2c}{1+\alpha}\cdot 2T^{-1}\left(\frac{2c}{1+\alpha}\right) \\
&\geq 0.
\end{aligned}$$

Thus, $g(\alpha)$ is increasing in α. Then, there exists a threshold value $\tilde{\alpha}(c)$ such that

$$\begin{cases} g(\alpha) \leq R\left[T^{-1}(c)\right], & if \quad \alpha < \tilde{\alpha}(c), \\ g(\alpha) = R\left[T^{-1}(c)\right], & if \quad \alpha = \tilde{\alpha}(c), \\ g(\alpha) \geq R\left[T^{-1}(c)\right], & if \quad \alpha > \tilde{\alpha}(c). \end{cases}$$

Note that $c \leq \frac{1+\alpha}{2}$ when $\Pi_W \geq \Pi_N$, then we have $\tilde{\alpha}(c) \geq 2c-1$.
(b) For any fixed $\alpha \in [0,1]$, if $c \geq \frac{1+\alpha}{2}$, $\Pi_W = 0 \leq \Pi_N$. Thus, there exists a $\overline{c}(\alpha) \leq \frac{1+\alpha}{2}$ such that $\Pi_W \leq \Pi_N$ if $c \geq \overline{c}(\alpha)$.
Besides, $R\left[T^{-1}(c)\right]$ is decreasing in c, so that

$$R\left[T^{-1}(c)\right] \leq R\left[T^{-1}(0)\right] = \mathbb{E}(S).$$

As $(1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right]$ is also decreasing in c, we have

$$(1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right] \geq \mathbb{E}(S) \quad when \quad c \leq \frac{1+\alpha}{2}T\left[R^{-1}\left(\frac{\mathbb{E}(S)}{1+\alpha}\right)\right].$$

Thus, there exists a $\underline{c}(\alpha) \geq \frac{1+\alpha}{2}T\left[R^{-1}\left(\frac{\mathbb{E}(S)}{1+\alpha}\right)\right]$ such that $\Pi_W \geq \Pi_N$ if $c \leq \underline{c}(\alpha)$. □

B.3. **Proof of Proposition 6.3.**

Proof. When $\alpha = \tilde{\alpha}(c)$, $\Pi_W = \Pi_N$, then we have

$$\left[1+\tilde{\alpha}(c)\right]R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right] = R\left[T^{-1}(c)\right].$$

Differentiating both sides of the above equation with respect to c, we obtain that

$$\frac{\mathrm{d}Right}{\mathrm{d}c} = R^{'}\left[T^{-1}(c)\right]\left(T^{-1}\right)^{'}(c) = \frac{R^{'}\left[T^{-1}(c)\right]}{T^{\prime}\left[T^{-1}(c)\right]} = -2T^{-1}(c),$$

$$\begin{aligned}
\frac{\mathrm{d}Left}{\mathrm{d}c} =& \left[1+\tilde{\alpha}(c)\right]R^{'}\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]\left(T^{-1}\right)^{'}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\frac{2\left[1+\tilde{\alpha}(c)\right]-2c\tilde{\alpha}^{'}(c)}{\left[1+\tilde{\alpha}(c)\right]^2} \\
&+ \tilde{\alpha}^{'}(c)R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right] \\
=& \frac{R^{'}\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]}{T^{\prime}\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]} \cdot \frac{2\left[1+\tilde{\alpha}(c)\right]-2c\tilde{\alpha}^{'}(c)}{1+\tilde{\alpha}(c)} + \tilde{\alpha}^{'}(c)R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right] \\
=& \frac{2T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\int_{T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)}^{B}\frac{1}{s}f(s)\mathrm{d}s}{-\int_{T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)}^{B}\frac{1}{s}f(s)\mathrm{d}s} \cdot \frac{2\left[1+\tilde{\alpha}(c)\right]-2c\tilde{\alpha}^{'}(c)}{1+\tilde{\alpha}(c)} \\
&+ \tilde{\alpha}^{'}(c)R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right] \\
=& -2T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right) \cdot \left[2 - \frac{2c}{1+\tilde{\alpha}(c)}\tilde{\alpha}^{'}(c)\right] + \tilde{\alpha}^{'}(c)R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]
\end{aligned}$$

$$=\tilde{\alpha}^{'}(c)\left\{R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]+\frac{4c}{1+\tilde{\alpha}(c)}\right\}-4T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right).$$

As $\frac{\mathrm{d}Left}{\mathrm{d}c}=\frac{\mathrm{d}Right}{\mathrm{d}c}$, we have

$$\tilde{\alpha}^{'}(c)\left\{R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]+\frac{4c}{1+\tilde{\alpha}(c)}\right\}=4T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)-2T^{-1}(c).$$

Note that $\tilde{\alpha}(c)\geq 0$, so

$$R\left[T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\right]+\frac{4c}{1+\tilde{\alpha}(c)}>0.$$

Thus, we have

$$\begin{aligned}\tilde{\alpha}^{'}(c)\leq 0 \quad &\Longleftrightarrow \quad 4T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)-2T^{-1}(c)\leq 0,\\ &\Longleftrightarrow \quad T^{-1}\left(\frac{2c}{1+\tilde{\alpha}(c)}\right)\leq\frac{1}{2}T^{-1}(c),\\ &\Longleftrightarrow \quad \frac{2c}{1+\tilde{\alpha}(c)}\geq T\left[\frac{1}{2}T^{-1}(c)\right],\\ &\Longleftrightarrow \quad \tilde{\alpha}(c)\leq\frac{2c}{T\left[\frac{1}{2}T^{-1}(c)\right]}-1.\end{aligned}$$

Let $\alpha=\frac{2c}{T\left[\frac{1}{2}T^{-1}(c)\right]}-1$,

$$\begin{aligned}\tilde{\alpha}(c)\leq\alpha \quad &\Longleftrightarrow \quad \Pi_N\leq\Pi_W,\\ &\Longleftrightarrow \quad R\left[T^{-1}(c)\right]\leq(1+\alpha)R\left[T^{-1}\left(\frac{2c}{1+\alpha}\right)\right],\\ &\Longleftrightarrow \quad R\left[T^{-1}(c)\right]\leq\frac{2c}{T\left[\frac{1}{2}T^{-1}(c)\right]}\cdot R\left[\frac{1}{2}T^{-1}(c)\right].\end{aligned}$$

Then, we have

$$\tilde{\alpha}^{'}(c)\leq 0 \quad \Longleftrightarrow \quad 2c\cdot R\left[\frac{1}{2}T^{-1}(c)\right]\geq T\left[\frac{1}{2}T^{-1}(c)\right]\cdot R\left[T^{-1}(c)\right].$$

□

REFERENCES

[1] R. Albuquerque, Y. Koskinen and C. Zhang, Corporate Social Responsibility and Firm Risk: Theory and Empirical Evidence, *Management Science*, **65(10)** (2019), 4451-4469.
[2] A. Arya and B. Mittendorf, Supply chain consequences of subsidies for corporate social responsibility, *Production and Operations Management*, **24** (2015), 1346-1357.
[3] L. Barcos, A. Barroso, J. Surroca and J. A. Tribo, Corporate social responsibility and inventory policy, *International Journal of Production Economics*, **143** (2013), 580-588.
[4] M. Besiou and L. N. Van Wassenhove, Addressing the challenge of modeling for decision-making in socially responsible operations, *Production and Operations Management*, **24** (2015), 1390-1401.
[5] C. M. Chen and M.Delmas, Measuring corporate social performance: An efficiency perspective, *Journal of Marketing*, **61(1)** (1997), 68-84.
[6] L. Chen and S. S. Yang, Are purchase-triggered donations advantageous with competition?, *Journal of the Operational Research Society*, **68:3** (2017), 237-252.

[7] L. Y. Chu, G. Li and P. Rusmevichientong, Optimal pricing and inventory planning with charitable donations, *Manufacturing & Service Operations Management*, **20(4)** (2018), 687-703.

[8] J. Cong, T. Pang and H. Peng, Optimal strategies for green supply chain considering social responsibility and environmental responsibility, *Journal of Industrial and Management Optimization*, **19(12)** (2023), 8483-8504.

[9] L. Dai, T. Shu, Y. Zheng, S. Chen and S. Wang, Impacts of reference price effect and corporate social responsibility on the pricing strategy of a remanufacturing supply chain, *Journal of Industrial and Management Optimization*, **19(10)** (2023), 7350-7379.

[10] R. G. Dugan, J. J. Clarkson and J. T. Beck, When cause-marketing backfires: differential effects of one-for-one promotions on hedonic and utilitarian products, *Journal of Consumer Psychology*, **31** (2021), 532-550.

[11] K. Elsayed, Corporate social responsibility: Does inventory management matter, *International Journal of Supply Chain and Operations Resilience*, **1(2)** (2015), 139-156.

[12] R. Fisman, G. Heal and V. B. Nair, A model of corporate philanthropy, New York: Columbia University and University of Pennsylvania, 2006.

[13] F. Gao, Cause marketing: product pricing, design, and distribution, *Manufacturing & Service Operations Management*, **22(4)** (2020), 775-791.

[14] X. Guo, P. Kouvelis and D. Turcic, Pricing, quality, and stocking decisions in a manufacturer-centric dual channel, *Manufacturing & Service Operations Management*, **24(4)** (2022), 2116-2133.

[15] A. Hamby, One for me, one for you: cause-related marketing with buy-one give-one promotions, *Psychology & Marketing*, **33(9)** (2016), 692-703.

[16] D. Hildebrand, Y. DeMotta, S. Sen and A. Valenzuela, Consumer responses to corporate social responsibility (CSR) contribution type, *Journal of Consumer Research*, **44(4)** (2017), 738-758.

[17] G. Iyer and D. A. Soberman, Social responsibility and product innovation, *Marketing Science*, **35(5)** (2016), 727-742.

[18] M. Isaza and L. Italie, 60 million pairs of shoes and 10 years later: Toms Shoes founder reflects, *Seattle Times*, May 13, 2016. Available from: https://www.seattletimes.com/life/fashion/60-million-pairs-of-shoes-and-10-years-later-toms-shoes-founder-reflects/.

[19] A. Krishna and U. Rajan, Cause marketing: spillover effects of cause-related products in a product portfolio, *Management Science*, **55(9)** (2009), 1469-1485.

[20] X. Li, C. Zhang, R. Lyu and Z. Li, Direct-store-delivery strategy of a supply chain considering corporate social responsibility and pre-sales service under demand uncertainty, *Journal of Industrial and Management Optimization*, **20(5)** (2024), 1719-1771.

[21] P. Letizia and G. Hendrikse, Supply chain structure incentives for corporate social responsibility: An incomplete contracting analysis, *Production and Operations Management*, **25(11)** (2016), 1919-1941.

[22] J. D. Lowrey, T. J. Richards and S. F. Hamilton, Food Donations, Retail Operations, and Retail Pricing, *Manufacturing & Service Operations Management*, **25(2)** (2023), 792-810.

[23] C. Marquis and A. Park, Inside the buy-one give-one model, *Stanford Social Innovation Review*, **12** (2014), 28-33.

[24] M. Mahmoudzadeh and A. Chaturvedia, A behavioral model of responsible sourcing in supply chains: the role of dual-sourcing bias, *Decision Analysis*, (2024).

[25] H. S. Naeeni, F. Sahin and E. P. Robinson Jr., Socially responsible product-positioning: Impact of halo/horns spillover on product image, *European Journal of Operational Research*, **308** (2023), 852–863.

[26] A. Ozbilge, E. Hassini and M. Hassini, Optimal pricing and donation policy for fresh goods, *European Journal of Operational Research*, **312** (2024), 198-210.

[27] S. Park, S. Ahn and S. Kim, Sport event vs. art event? The effect of CSR-linked sponsorship event type on product purchase, *Journal of Retailing and Consumer Services*, **77** (2024), 103683.

[28] S. Park, W. T. Huh and B. C. Kim, Optimal inventory management with buy-one-give-one promotion, *IISE Transactions*, **54:2** (2022), 198-209.

[29] K. Sahlin-Andersson, Corporate social responsibility: a trend and a movement, but of what and for what, *Corporate Governance Internat. J. Bus. Soc.*, **6(5)** (2006), 595-608.

[30] S. Sen and C. B. Bhattacharya, Does doing good always lead to doing better? Consumer reactions to corporate social responsibility, *Journal of Marketing Research*, **38(2)** (2001), 225-243.

[31] P. R. Varadarajan and A. Menon, Cause-related marketing: a coalignment of marketing strategy and corporate philanthropy, *Journal of Marketing*, **52(3)** (1988), 58-74.

[32] M. Xu and X. Li, The interplay between e-tailer information sharing and supplier cause marketing, *International Journal of Production Research*, **60(12)** (2022), 3863-3878.

[33] Y. Yang and Y. Jiang, The impact of suppliers' CSR controversies on buyers' market value: The moderating role of social capital, *Journal of Purchasing and Supply Management*, **30** (2024), 100904.

[34] P. Y. Yen, Y. J. Chen and Q. C. He, A better buy-one-give-one business model, *Naval Research Logistics*, **68** (2021), 1070-1081.

[35] Z. Zhang, D. Ren, Y. Lan and S. Yang, Price competition and blockchain adoption in retailing markets, *European Journal of Operational Research*, **300** (2022), 647-660.

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