Journal of Industrial and Management Optimization
doi:10.3934/jimo.2025171

VOTER MODEL AND EQUILIBRIUM FOR COOPERATIVE ADVERTISING IN SOCIAL NETWORK

Zhe Yu[1], Yisheng Yang[2] and Jinxing Xie[*1]

[1]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

[2]School of Mathematical Sciences, Beihang University, Beijing 100191, China

(Communicated by Shuhua Zhang)

Abstract. This paper examines a cooperative advertising game between a manufacturer and a retailer within a social network. The manufacturer first decides the participation rate for the retailer, and then the retailer decides the advertising expenditures for consumers in the social network. To describe the features of cooperative advertising through social media, we adopt the voter model commonly used to delineate communication among social media users. Using this model, we calculate the expected profits of both the manufacturer and retailer, and propose an algorithm to identify the Stackelberg equilibrium of their strategies. We also present an improved algorithm that simplifies the process of finding the optimal participation rate for the manufacturer. Numerical experiments are conducted to show the impact of model parameters on the strategies and profits of the manufacturer and retailer, as well as the profitability of the cooperative advertising scheme, revealing that cooperative advertising significantly improves the profits of the manufacturer and retailer under various contexts where consumers communicate in a social network.

1. **Introduction.** Cooperative advertising (co-op ad) is a prevalent cost-sharing scheme in business between a manufacturer and a retailer, where the manufacturer subsidizes a portion of the retailer's advertising expenses, thereby spurring the retailer to bolster its advertising endeavors to increase sales. The practice of cooperative advertising dates back to the 1930s [25], and the reasons for its popularity have been extensively studied in literature ([19], [40], and among others). For example, Hutchins [19] offers a justification for the prevalence of cooperative advertising by differentiating between global advertising conducted by the manufacturer and local advertising led by the retailer. While global advertising attempts to build a brand's goodwill but does not necessarily lead to consumers' purchasing actions, local advertising aims at stimulating short-term and immediate sales. Without a cost-sharing scheme, the retailer usually does not advertise to the necessary level as the manufacturer expects, thus it benefits both the manufacturer and retailer if the manufacturer shares a portion of the retailer's local advertising expenditures. The shared percentage is called the manufacturer's *participation rate* and acts as

2020 *Mathematics Subject Classification.* Primary: 90B60, 91A80; Secondary: 90B10, 91A10.
Key words and phrases. Cooperative advertising, social network, voter model, Stackelberg equilibrium, optimization.

This work was supported by the National Natural Science Foundation of China (NSFC) [Grant No. 71671099].

*Corresponding author: Jinxing Xie.

a key parameter in the cooperative advertising scheme. In addition, the fact that local advertising is usually much more economical than global advertising is also a possible reason [40].

The 21st century has witnessed an explosive expansion in the extent and depth of cooperative advertising. According to a report in 2018 [36], the market value of cooperative advertising has ballooned to surpass $70 billion annually in the United States. Moreover, cooperative advertising tactics have adapted to meet the challenges and opportunities of the digital age. Mosello [30] observed that the cooperative advertising market in North America in 2015 was estimated to be $36 billion, with an impressive 69% of this sum allocated to fund digital marketing initiatives. This is quite understandable, since with the popularity and rapid development of the mobile Internet and digital economics, the communication landscape between marketers and consumers has been significantly reshaped by the advent of social media. The vast swathes of data collected from social media platforms empower firms to precisely target their advertisements towards potential customers, and make payments based on the actual impact of the ads [17]. Another feature of social media advertising is the importance of communication between consumers as advertising channels. Modern social networks allow individuals to disseminate their ideas and opinions, thereby transforming each person from a passive recipient to an active conduit of information flow.

The development of modern social media has introduced new forms of cooperative advertising. For example, the Amazon vendor central co-op agreement [2] requires manufacturers to contribute market development funds (MDF), which Amazon then uses to carry out advertising activities such as running digital ads and organizing promotions. These initiatives preserve the essential features of cooperative advertising while adapting to modern social media by leveraging technologies such as personalized recommendations and microtargeting.

To adapt to the unique characteristics of social media, it is essential to incorporate social networks into cooperative advertising research. There are, however, very few studies on this topic. To the best of our knowledge, [23] is the only work available addressing this issue where a manufacturer sells through a retailer to consumers in a social network. The problem is modeled as a Stackelberg game, in which the manufacturer as the game leader, offers the retailer a cooperative advertising scheme with a participation rate, and then the retailer decides advertising expenditures to consumers. The total demand of all consumers in the social network depends not only on the advertisements allocated to them, but also on the structure of the social network. In particular, the amount consumed by each individual consumer is derived from their utility which is influenced by the amount consumed by themselves as well as their peers, without analyzing the communication processes among consumers in the network.

Our research explores a similar problem as in [23] but from a new perspective by explicitly delving into the communication and imitation processes among consumers. We aim to answer the following questions: 1) How can the total demand of consumers in the social network be quantified when the communication and imitation processes are taken into consideration? 2) Is there any efficient algorithm to calculate the Stackelberg equilibrium strategies (i.e., the optimal participation rate which should be announced by the manufacturer, and the optimal advertising expenditures which should be allocated to consumers by the retailer)? 3) How are the profits of the manufacturer and retailer affected by model parameters and the cooperative advertising scheme?

The rest of the article is organized as follows. We will review related literature, point out the research gap, and highlight our contributions in more detail in Section 2. Section 3 presents the modeling framework. Sections 4 and 5 provide a basic algorithm and an improved algorithm to solve for the Stackelberg equilibrium strategies, respectively. With numerical illustrations, Section 6 presents the impact of model parameters on the strategies and profits of the manufacturer and retailer, while Section 7 shows the profitability of the cooperative advertising scheme. Section 8 concludes the paper by summarizing key contributions and managerial insights, acknowledging its limitations, and suggesting directions for future research.

2. **Literature review.**

Our research is closely related to cooperative advertising, and communication and advertising in social networks. In the following, we briefly review these two streams of literature, and summarize the research gaps to reveal the contributions of our research.

2.1. **Cooperative advertising.**

A critical issue in cooperative advertising is to make decisions on the optimal participation rate provided by the manufacturer and the retailer's ideal advertising allocation. There is a vast literature on this topic, with most of the theoretical and modeling research adopting game theory models. Depending on whether the decisions evolve with time, these models can be divided into two categories: static and dynamic. Two exhaustive reviews of these models and discussions up to 2014 have been provided in [3] and [20].

In the last decade, there are many extensions in cooperative advertising involving e-tailing platforms, channel selection, and competition dynamics, etc. Li et al. [22] integrated the manufacturer's encroachment and the retailer's behavioral preference into cooperative advertising, and found that retailer's fairness concerns significantly impact the decisions of the manufacturer and retailer. Yang et al. [38] investigated the effects of cooperative adverting in a dual channel, where a national brand manufacturer sells online as well as through a retailer's channel, and the retailer sells both the national brand and his store brand. They adopted the commonly-used square root functions as the demand responses with respect to the advertising expenditures, and compared the supply chain members' decisions and profits under the centralized case and four decentralized gaming scenarios. Chaab et al. [8] extended the dual-channel cooperative advertising framework by factoring in channel preference and web compatibility considerations. Wu et al. [37] integrated asymmetric information into the dual-channel cooperative advertising. Their findings indicate that manufacturers benefit from information sharing, and that manufacturers can foster such information sharing by proposing higher participation rates. Karray et al. [21] considered a distribution channel with two competitive manufacturers and two competitive retailers selling both products. They identified the closed-form Stackelberg equilibrium for the competitive manufacturers and retailers, and concluded that cooperative advertising is profitable only when store and product competition is relatively low, or consumers are highly differentiated. Han et al. [15] examined the market cycle's influence on advertising outcomes. Utilizing a Markov chain to represent the market cycle, they employed game theory to elucidate how channel members adopt a forward-looking advertising strategy in response to market fluctuations. Huang and Yan [18] studied the cooperative promotion strategies between a manufacturer and a retail platform. While both of them make their decisions on the advertising efforts, the cooperation between them is considered as a cost-sharing scheme for the sales promotion such as price discounts, instead of advertisements.

2.2. **Communication and advertising in social networks.** Academic interest in social networks dates back more than fifty years. Major research problems include information dissemination within social networks, advertising decisions, and consumer–producer interactions. Regarding information dissemination, Holley and Liggett [16] employed a graph to illustrate communication patterns within a social network. Aldous and Fill [1] investigated information dissemination in a connected social network and concluded that a society with only binary attitudes will eventually arrive at a consensus. Mobilia [28] and Mobilia et al. [29] introduced the notion of a "zealot" or "stubborn" consumer into social network research, i.e., an individual who can influence others but remains impervious to input from neighbors or attitude alterations. By integrating the idea of stubborn consumers, it becomes possible to include advertisers within the social network and analyze advertising decisions and effects. Bagarello et al. [4] employed a ρ-Hamilton method to analyze how real and fake news spread within a social network composed of different types of agents. Chen et al. [10] examined the influence of opinion leaders in social networks. Using dynamic opinion theory and simulation, they concluded that the weight of advertising influence has a dual effect and should be maintained within an effective range. Meng et al. [26] developed a model to describe information-spreading dynamics, finding that the highly clustered nature of social networks results in high-frequency information bursts with relatively limited coverage, thereby enabling social media to support high capacity and diversity in information dissemination.

With the rapid rise in social media users, the past two decades have seen a growing body of literature on social media advertising [32]. Zubcsek and Sarvary [41] examined how a firm's advertising activity is influenced by the size and connectivity of the social network. Yildiz et al. [39] employed the dual method proposed in [11] to analyze social networks with stubborn consumers, showing that the average attitude in such a network will achieve distributional convergence. Goyal et al. [13] studied advertising competition in a social network with binary attitudes, identifying the Nash equilibrium of competitors. They also defined the price of anarchy (PoA) to describe the loss of efficiency in the supply chain attributed to competition. Qian et al. [34] discussed a makeshift effect of advertising, where advertising for new products through e-retailers also benefits independent remanufacturers selling previous generations of goods.

In addition to the information-spreading process, another major issue in social networks is producer–consumer behavior. Bloch and Quérou [6] examined a monopoly pricing scenario in which consumers' utility is influenced by comparisons between their own consumption and that of their neighbors. They demonstrated that producers typically do not employ price discrimination in a normal network, but this network irrelevance disappears when quadratic costs and direct influence are introduced. In such cases, producers tend to charge consumers with higher centrality a higher price. Chen et al. [9] extended the consumer behavior problem to a duopoly competition game, considering network externalities, i.e., consumers are motivated to conform to their peers. They construct a duopoly pricing game to explore how advertisers leverage these externalities to compete for market share and optimize profitability. On the other hand, Cao et al. [7] focused on a negative-externality problem, where purchasing the same product as neighbors diminishes the consumer's utility. Negative externalities align with the characteristics of certain specific markets, such as clothing and luxury markets. Based on assumptions of impatient consumers and complete information, they discussed the optimal pricing strategy, proved that the problem is NP-hard, and provided a 2-approximation algorithm to find the optimal pricing solution.

In this article, we use the voter model to describe the cooperative advertising problem in social networks. The voter model is widely used in social network analysis. According to Liggett [24], a voter model consists of a social network in which each node can adopt one of two opinions. Additionally, the attitude of each node is influenced by its neighbors, making it an appropriate tool for studying advertisement diffusion in a social network. Bimpikis et al. [5] applied the voter model to analyze advertising competition within a social network. They considered consumer attitudes toward purchasing a product as ranging continuously from 0 to 1 and introduced an additional opinion, status quo, which will be used to examine advertisement resistance in our article. They showed that, given a connected social network and a constant advertising allocation, the average consumer attitude will eventually converge. Recent studies on the voter model include Meyer and Metzler [27], who used a voter model with a number of zealot agents to analyze shifts in political opinion within a society. Ramirez et al. [35] also explored order dynamics, but with different communication rates. They employed a nonlinear voter model where an individual adopts an opinion scale based on the qth power of the number of neighbors. They find that dynamics differ depending on the value of q: When $q > 1$, the voter model favors majority opinions, while when $q < 1$, it promotes minority ideas.

2.3. **Research gap and contributions.** Our research helps fill the research gap in combining social networks with cooperative advertising. As far as we know, [23] is the only literature that addresses this problem. However, our study builds on a different research foundation. In [23], Liang et al. used a utility-based approach to determine consumer demand, assuming that the product is infinitely divisible and that consumers can purchase any quantity they desire. This approach is suitable for markets such as food, water, and wholesale goods.

Our study applies to a different type of market. Based on our voter model, we assume that the product is indivisible, and each consumer makes a binary purchasing decision ("YES" or "NO"), meaning that if they decide to purchase, they buy only one unit of the product. Under this assumption, the total market size is constrained by the size of the social network, and advertisers use advertisements to capture the market, rather than expanding it. These assumptions reflect the nature of markets such as consumer electronics and automobiles, where consumers typically purchase only one smartphone or car at a time. In these markets, market share becomes the primary concern, and companies advertise to build their brand and capture a portion of the consumers. Furthermore, the mechanisms used to determine demand and advertising costs differ from those in [23]. The voter model clearly delineates the spread of advertisements among consumers, allowing us to track each consumer's attitude. Thus, rather than using utility, the expected demand is determined by the aggregate attitude of the consumers.

In addition, the advertising cost function used in [23] is borrowed from traditional advertising modes, while in this paper we will adopt the cost-per-click (CPC) payment mechanism, which is popular in Internet advertising, to reflect the feature of digital advertisements in networks [17].

The contributions of this research in addressing the identified research gap can be summarized as follows: 1) Building on the voter model, we develop a mathematical framework for cooperative advertising in a social network that incorporates both consumer interactions and the features of digital advertising in social media. 2) We solve the Stackelberg game to determine the optimal strategies of the manufacturer and retailer. In particular, we characterize several properties of these strategies, such as the convexity of the manufacturer's objective function in the Stackelberg game, and construct an algorithm to compute the optimal participation rate. 3) Based on the problem's properties such as monotonicity, we formulate a series of surrogate problems to design an improved algorithm that computes the optimal participation rate and advertisement allocation more efficiently. 4) We analyze how the manufacturer and retailer respond to varying market conditions and validate the profitability of cooperative advertising.

3. **Model setup.** In this section, we first derive the total demand of consumers in the social network, and then present the game model for cooperative advertising.

3.1. **Consumers' communications and purchasing quantities.** Consider a distribution channel consisting of a manufacturer (M) and a retailer (R). R sells M's product to n potential consumers in a social network $G = (N, E)$, where $N = \{1, 2, ..., n\}$ is the set of consumers, and $E = (e_{ij})_{n\times n} \subseteq [0, 1]^{n\times n}$ is the *communication matrix*, a matrix indicating the frequencies of communications between consumers. More specifically, when consumer $i \in N$ receives a piece of information from one of their neighbors, this information is from consumer $j \in N$ with probability e_{ij}. Assume $e_{ii} = 0, e_{ij} \geq 0$ $(i \neq j)$, and $\sum_{j=1}^{n} e_{ij} = 1$ for all i.

Each of the potential consumers purchases at most one unit of the product, and the probability to purchase is measured by their attitude toward the product. The attitude evolves over time depending on the advertisement information received from the retailer and the communications between consumers. We use the voter model proposed in [5] to delineate the communications among consumers, and derive the demand of the product based on their final attitude.

In the voter model, the communication processes are cut into multiple periods. In each period, each consumer receives a piece of information either from one of their neighbors or from the retailer's advertisement, and updates their attitude according to the information received. Denote $\beta \in (0, 1)$ as the vitality of the social network, i.e., in each period, a consumer receives the message from one of his neighbors with probability β. Let $\alpha = 1 - \beta$ be the advertisement sensitivity of the social network, i.e., a consumer receives information from the retailer's advertisement with probability α.

Define $m_i[k] \in [0, 1]$ as consumer i's attitude towards the product at period k, which means if the consumer makes the decision at period k, they will purchase the product with probability $m_i[k]$. At period 0, each consumer has a random initial attitude $m_i[0]$. At each period $k \geq 1$, the attitude is updated as follows:

$$m_i[k] = \frac{k-1}{k} m_i[k-1] + \frac{1}{k}\mathfrak{I}(a_i[k] = 1), \tag{1}$$

where $\mathfrak{I}$ is the indicator function, and $a_i[k]$ is the information received by the consumer i in period k, with $a_i[k] = 1$ meaning the information is in favor of the product. A constant $\lambda \geq 0$ captures the advertisement resistance of consumers. If the information is received from the retailer's advertisement, then $a_i[k] = 1$ with probability $x_i/(x_i+\lambda)$, where $x_i \geq 0$ is the retailer's advertisement expenditure spent on consumer i. If the information is received through the communication with a neighbor, say j, then $a_i[k]=1$ with probability $m_j[k]$.

In [5], it was proved that the consumers' attitudes will converge to a final status almost surely. Noticing that consumers' purchasing decisions are made based on their final attitudes, the total expected demand for the product can be expressed as follows:

$$d=\sum_{i=1}^{n} d_i=\alpha \sum_{i=1}^{n} c_i \frac{x_i}{x_i+\lambda}, \tag{2}$$

where c_i is the *absorption centrality* and $d_i=\alpha c_i x_i/(x_i+\lambda)$ is the expected purchasing quantity of consumer i. The absorption centrality vector c is defined as

$$c=(c_1,c_2,\ldots,c_n)=\mathbf{1}^T[I-(1-\alpha)E]^{-1}, \tag{3}$$

where $\mathbf{1}^T$ represents a row vector with all entries being ones, and I is the $n\times n$ identity matrix. From the definition, it is clear that $c_i \geq 1$.

3.2. **Game model for cooperative advertising.** Assume that R is the only advertiser in the social network and M does not advertise directly in the social network. M adopts the cooperative advertising program, announcing a participation rate $t\in[0,1)$ to R, which means M promises to reimburse a proportion t of R's advertisement cost.

Assume R advertises through an online advertising provider, and they adopt the widely-used CPC payment mechanism, which means R's advertisement cost depends on how many consumers receive and click on the advertisement [17]. When R's advertisement expenditure on consumer i is x_i, the expected actual advertisement cost for R is αx_i, since, as introduced in Section 3.1, consumer i reads the advertisement with probability α.

We model the interaction between M and R as a Stackelberg game with M being the leader and R being the follower. The sequence of events is as follows. First, M offers a participation rate $t\in[0,1)$ to R. Second, R decides the advertisement expenditure spent on each consumer i, which is denoted by x_i. Finally, the consumers decide on whether to purchase the product, M and R earn the profit, and M pays the advertisement subsidy to R according to the participation rate.

Furthermore, assume R's gross profit margin is p and M's gross profit margin is q, i.e., for each unit of product sold, R earns a profit of p and M earns a profit of q if the advertisement cost is not considered. Under these assumptions and using Equation (2), the expected profits of M and R are, respectively,

$$\pi_M=qd-t\sum_{i=1}^{n}\alpha x_i=\alpha q\sum_{i=1}^{n} c_i\frac{x_i}{x_i+\lambda}-\alpha t\sum_{i=1}^{n} x_i, \tag{4}$$

$$\pi_R=pd-(1-t)\sum_{i=1}^{n}\alpha x_i=\alpha p\sum_{i=1}^{n} c_i\frac{x_i}{x_i+\lambda}-\alpha(1-t)\sum_{i=1}^{n} x_i. \tag{5}$$

4. **Stackelberg equilibrium.** In this section, we discuss the properties of the Stackelberg equilibrium of the game and then provide an algorithm to compute the equilibrium.

4.1. **The special case when all consumers are active.** We solve the Stackelberg game with backward induction, first considering R's best response to the participation rate announced by M. For a given participation rate $t \in [0,1]$, R aims to maximize π_R in Equation (5). Noticing that π_R can be rewritten as $\pi_R(x|t) = \sum_{i=1}^{n} \pi_{Ri}(x_i|t)$, where

$$\pi_{Ri}(x_i|t) = pd_i - (1-t)\alpha x_i = \alpha p c_i \frac{x_i}{x_i+\lambda} - \alpha(1-t)x_i \tag{6}$$

represents R's expected profit earned from consumer i, maximizing π_R is equivalent to maximizing $\pi_{Ri}(x_i|t)$ for each consumer i.

It is straightforward to verify that the second derivative of $\pi_{Ri}(x_i|t)$ concerning x_i is negative. Therefore, R's optimal advertisement targeting on consumer i can be obtained through the first-order condition as

$$x_i(t) = \begin{cases} \sqrt{\dfrac{p\lambda c_i}{(1-t)}} - \lambda, & \text{if} \quad t \geq 1 - pc_i/\lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The following theorem is evident according to Equation (7).

Theorem 4.1. *The retailer's best response $x_i(t)$ is characterized by Equation (7), which is non-decreasing in c_i, and is strictly increasing in c_i when $t \geq 1 - pc_i/\lambda$.*

Theorem 4.1 indicates that a consumer with a higher centrality c_i deserves higher advertisement expenditure $x_i(t)$ in the optimal strategy of R.

For convenience, when $x_i(t) > 0$, we say that consumer i is "*active*", otherwise, it is "*non-active*". We define the "*active set*" as the set of all active consumers. We start with a special case when all consumers are active in this subsection, and consider the more general case in Section 4.2. A sufficient condition for this special case is $c_i > \lambda/p$ for all i, which ensures $x_i(t) > 0$ for all i and $t \in [0,1)$.

When all consumers are active, substituting Equation (7) into Equation (6), the expected profit of R earned from consumer i is

$$\pi_{Ri}(x_i(t)|t) = \alpha \left[pc_i + (1-t)\lambda - 2\sqrt{(1-t)p\lambda c_i} \right] = \alpha \left(\sqrt{pc_i} - \sqrt{(1-t)\lambda} \right)^2. \tag{8}$$

Similarly, π_M in Equation (4) can be rewritten as

$$\pi_M(x,t) = \sum_{i=1}^{n} \pi_{Mi}(x_i,t), \tag{9}$$

where

$$\pi_{Mi}(x_i,t) = qd_i - \alpha t x_i = \alpha q c_i \frac{x_i}{x_i+\lambda} - \alpha t x_i \tag{10}$$

represents the expected profit of M earned from consumer i.

Substituting Equation (7) into Equation (10), we obtain

$$\pi_{Mi}(t) \equiv \pi_{Mi}(x_i(t),t) = \alpha \left[qc_i + t\lambda - \left(\sqrt{\frac{q^2}{p}} + \frac{\sqrt{p}t}{1-t} \right) \sqrt{(1-t)\lambda c_i} \right]. \tag{11}$$

The expected profits earned by R and M are, respectively,

$$\begin{aligned} \pi_R(t) &\equiv \pi_R\left(x(t)|t\right) \\ &= \sum_{i=1}^{n} \pi_{Ri}\left(x_i(t)|t\right) \\ &= \alpha\left[(pC + (1-t)n\lambda) - 2\sqrt{(1-t)p\lambda}T\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} \pi_M(t) &\equiv \pi_M\left(x(t), t\right) \\ &= \sum_{i=1}^{n} \pi_{Mi}\left(t\right) \\ &= \alpha\left[(qC + n\lambda t) - \left(\sqrt{\frac{q^2}{p}}\sqrt{1-t} + \frac{\sqrt{p}t}{\sqrt{1-t}}\right)\sqrt{\lambda}T\right], \end{aligned} \tag{13}$$

where

$$C = \sum_{i=1}^{n} c_i, \; T = \sum_{i=1}^{n} \sqrt{c_i}. \tag{14}$$

We now discuss how M optimizes the participation rate t. The first-order derivative of $\pi_M(t)$ is

$$\frac{d\pi_M(t)}{dt} = \frac{\alpha T\sqrt{\lambda}}{2}(1-t)^{-\frac{3}{2}}\left[2\left(\frac{n\sqrt{\lambda}}{T}\right)(1-t)^{\frac{3}{2}} - \frac{1}{\sqrt{p}}\left((q-p)t - (q-2p)\right)\right], \tag{15}$$

and the second-order derivative is

$$\frac{d^2\pi_M(t)}{dt^2} = -\frac{\alpha T\sqrt{\lambda}}{4\sqrt{p}}(1-t)^{-\frac{5}{2}}\left((q-p)t - (q-4p)\right). \tag{16}$$

Throughout this article, we assume that $q < 4p$, which is not uncommon in modern industries, particularly in the consumer electronics and automobile markets. According to a report from GuruFocus News [14], Xiaomi, one of the world's leading smartphone manufacturers, achieved a gross profit margin of 12.6% in fiscal year 2024. Industry research by Ouzts [31] further indicates that smartphone retailers in the United States typically operate with gross profit margins of 5%–10%. In the automobile market, it is normal that the average gross profit margin for car manufacturers is 10% [33], while BusinessDojo [12] estimates that car retailers generally achieve gross profit margins of 5%–8%. In both industries, the condition $q < 4p$ holds strictly, supporting the validity of our assumption.

Under this assumption, it is not difficult to verify that $d^2\pi_M(t)/dt^2 \leq 0$ for $t \in [0, 1)$, indicating that $\pi_M(t)$ is a concave function of t.

Now, consider the first-order condition $d\pi_M(t)/dt = 0$, i.e.,

$$2\left(\frac{n\sqrt{\lambda}}{T}\right)(1-t)^{\frac{3}{2}} = \frac{1}{\sqrt{p}}\left((q-p)t - (q-2p)\right). \tag{17}$$

Noticing that $\lim_{t\to 1} d\pi_M(t)/dt = -\infty$, Equation (17) has a unique solution in $(0, 1)$ if and only if $d\pi_M(t)/dt|_{t=0} > 0$, i.e.,

$$\frac{2n\sqrt{\lambda}}{T} > \frac{(2p-q)}{\sqrt{p}}. \tag{18}$$

Theorem 4.2. *Suppose $q < 4p$ and all consumers are active. If inequality (18) holds, the manufacturer's optimal participation rate $t^* \in (0,1)$ is the unique solution of Equation (17); otherwise $t^* = 0$.*

For example, when M and R earn the same gross profit in the sale of one unit of product (i.e., $p = q$), the optimal participation rate is

$$t^* = \max\{1 - \sqrt[3]{\frac{pT^2}{4n^2\lambda}}, 0\}. \tag{19}$$

This equation implies that t^* is a decreasing function of p and an increasing function of λ, both in the non-strict sense.

4.2. **The general case.** Now we consider M's optimal participation rate in a more general case, i.e., relaxing the assumption that all consumers are active. In this case, for a participation rate $t \in [0,1)$, some consumers $i \in N$ may have

$$x_i(t) = \max\left\{\sqrt{\frac{p\lambda c_i}{(1-t)}} - \lambda, 0\right\} = 0. \tag{20}$$

We define the *activation rate* of consumer $i \in N$ as $t_i = 1 - pc_i/\lambda$, i.e., $x_i(t) > 0$ if and only if the participation rate $t \in (t_i, 1)$.

Without loss of generality, we assume that $c_1 \geq c_2 \geq \ldots \geq c_n$, meaning that the consumers are indexed according to their absorption centralities. From the perspective of activation rates, it is equivalent to $t_n \geq t_{n-1} \geq \ldots \geq t_1$.

For ease of exposition, we define $t_{n+1} = 1$ and $K = \min\{1 \leq i \leq n+1 | t_i \geq 0\}$. If $K = n+1$, then $c_i > \lambda/p$ for all $i \in N$, indicating that all the consumers are active for any $t \in [0,1)$ and the optimal participation rate t^* is characterized in Theorem 4.2. Hereafter, in this section, we assume $1 \leq K \leq n$.

Recall that an "active set" is the set of all active consumers. Given participation rate t, if consumer i is active, then consumer j $(j \leq i)$ is also active according to Theorem 4.1. Therefore, only the following active sets are possible:

$$S_k = \{1, 2, \ldots, k\}, \ k = K-1, \ldots, n, \tag{21}$$

where S_k is the active set with consumers $1, 2, \ldots, k$ being active and the others being non-active. Please note that S_{K-1} is an active set if and only if the participation rate $t \in [0, t_K]$, and $S_k (k \geq K)$ is an active set if and only if the participation rate $t \in (t_k, t_{k+1}]$.

Thus, the expected profit of M can be expressed as

$$\pi_M(t) = \begin{cases} \pi_M^{K-1}(t) \equiv \sum_{i=1}^{K-1} \pi_{Mi}(t), t \in [0, t_K], \\ \pi_M^k(t) \equiv \sum_{i=1}^{k} \pi_{Mi}(t), t \in (t_k, t_{k+1}], k = K, K+1, \ldots, n, \end{cases} \tag{22}$$

where $\pi_{Mi}(t)$ is defined as in Equation (10). This implies that $\pi_M(t)$ is a piecewise function with $n - K + 2$ segments and may not be differentiable at the endpoints t_k $(k = K-1, K, \ldots, n)$. However, $\pi_M(t)$ is a continuous function with respect to $t \in [0,1)$, and thus the interval $(t_k, t_{k+1}]$ in Equation (22) can be replaced with $[t_k, t_{k+1}]$.

The decision problem of M involves choosing an optimal participation rate t, which maximizes $\pi_M(t)$ in Equation (22). The optimal t can be obtained by solving the following $n-K+2$ subproblems:

$$\begin{aligned} &(\mathbf{SP}_{K-1}) \quad \max \ \pi_M^K(t) \ \ s.t. \ \ t \in [0, t_K], \\ &(\mathbf{SP}_k) \qquad \max \ \pi_M^k(t) \ \ s.t. \ \ t \in [t_k, t_{k+1}], k = K, K+1, \ldots, n. \end{aligned} \tag{23}$$

Note that each subproblem $\mathbf{SP}_k$ corresponds to a possible active set $S_k (k = K-1, \ldots, n)$. For the special case of $K = 1$, $S_{K-1} = S_0 = \emptyset$, therefore the subproblem $\mathbf{SP}_0$ has an objective function of zero and the optimal solution of $\mathbf{SP}_0$ can be denoted as $t^* = 0$. Below we consider the subproblem $\mathbf{SP}_k (k \geq 1)$.

By substituting the definition of $\pi_{Mi}(t)$ into Equation (23), we obtain

$$\pi_M^k(t) = \alpha \left[(qC_k + k\lambda t) - \left(\sqrt{\frac{q^2}{p}} \sqrt{1-t} + \frac{\sqrt{p}t}{\sqrt{1-t}} \right) \sqrt{\lambda} T_k \right], \tag{24}$$

where

$$C_k = \sum_{i=1}^{k} c_i, \ T_k = \sum_{i=1}^{k} \sqrt{c_i}, k = K-1, K, \ldots, n. \tag{25}$$

The first-order derivative of $\pi_M^k(t)$ is

$$\begin{aligned} &\frac{d\pi_M^k(t)}{dt} \\ &= \frac{\alpha T_k \sqrt{\lambda}}{2} (1-t)^{-\frac{3}{2}} \left[2 \left(\frac{k\sqrt{\lambda}}{T_k} \right) (1-t)^{\frac{3}{2}} - \frac{1}{\sqrt{p}} \left((q-p)t - (q-2p) \right) \right]. \end{aligned} \tag{26}$$

The corresponding first-order condition $d\pi_M^k(t)/dt = 0$ is

$$2 \left(\frac{k\sqrt{\lambda}}{T_k} \right) (1-t)^{\frac{3}{2}} = \frac{1}{\sqrt{p}} \left((q-p)t - (q-2p) \right). \tag{27}$$

Following the same arguments as in Section 4.1, it is not difficult to verify that $\pi_M^k(t) (k = K-1, K, \ldots, n)$ is a concave function of t under the assumption $q < 4p$. Therefore, each subproblem can be solved by finding the solution of Equation (27) and comparing the solution with the solutions at the endpoints t_k and t_{k+1}. Finally, by comparing the optimal solutions of all subproblems, the optimal participation rate t^* is obtained. This process is summarized in Algorithm 1.

Algorithm 1. Step 0. Relabel the consumers as $c_1 \geq c_2 \geq \ldots \geq c_n$. Calculate $t_i = 1 - \frac{pc_i}{\lambda} (i = 1, 2, \ldots, n)$, $t_{n+1} = 1$, and $K = \min\{1 \leq i \leq n+1 | t_i \geq 0\}$.

Step 1. If $K = n+1$, output t^* as in Theorem 4.2 and $\pi_M^* = \pi_M(t^*)$, then stop; otherwise, let $k = K-1$.

Step 2. If $k = 0$, let $\pi_M^* = 0$, $t^* = 0$, and proceed to Step 3. Otherwise, solve Equation (27) for subproblem $(\mathbf{SP}_{K-1})$ to obtain the solution s. If $s < 0$, then let $s = 0$; if $s > t_K$, then let $s = t_K$. Let $t^* = s$ and calculate $\pi_M^* = \pi_M^k(s)$.

Step 3. Let $k = k+1$. Solve Equation (27) for subproblem $(\mathbf{SP}_k)$ to obtain the solution s. If $s < t_k$, then let $s = t_k$; if $s > t_{k+1}$, then let $s = t_{k+1}$. Calculate $\pi_M^k(s)$. If $\pi_M^k(s) > \pi_M^*$, let $t^* = s$ and $\pi_M^* = \pi_M^k(s)$.

Step 4. If $k = n$, output t^* and π_M^*, then stop; otherwise, proceed to Step 3.

According to the previous arguments in this subsection, the following theorem holds.

Theorem 4.3. *Suppose $q < 4p$. The optimal participation rate t^* of M and the optimal expected profit π_M^* can be obtained with Algorithm 1.*

According to Theorem 4.1, it is easy to calculate the retailer's advertising expenditures $x_i(t^*)$ on consumers in the social network after the optimal participation rate t^* is obtained.

5. An improved algorithm and its efficiency.

In this section, we present an improved algorithm to calculate the Stackelberg equilibrium. We also demonstrate the improved algorithm's advantage in boosting efficiency.

5.1. An improved algorithm.

Algorithm 1 proposed in Section 4.2 requires solving $(n-K+2)$ subproblems and calculating $(n-K+2)$ expected profits. This can be time-consuming when n is large and K is small. In this subsection, we propose an improved algorithm which would be more efficient than Algorithm 1.

The manufacturer's decision problem for the participation rate is equivalent to the following optimization problem **P**:

$$\begin{aligned} &\max_{t\in[0,1)} \pi_M(t) = q\sum_{i=1}^{n} \alpha c_i \frac{x_i}{x_i+\lambda} - \alpha t \sum_{i=1}^{n} x_i, \\ &s.t.\ \ x_i = \max\left\{\sqrt{\frac{p\lambda c_i}{(1-t)}} - \lambda,\ 0\right\},\ i = 1,2,\ldots,n. \end{aligned} \tag{28}$$

We denote the optimal solution of **P** by $(t^*, x^*) \equiv (t^*, (x_i^*)_{i=1}^n)$. It is easy to see that (t^*, x^*) is a Stackelberg equilibrium of the cooperative advertising game model, where t^* is the manufacturer's optimal participation rate, and x_i^* is the retailer's advertising expenditure on consumer i.

Please note the similarities and the differences between **P** defined in (28) and $\mathbf{SP}_n$ defined in (23). It is easy to see that $\pi_M(t) = \pi_M^n(t)$ when $t \in [t_n, 1)$. However, problem **P** is defined for $t \in [0,1)$, while the subproblem $\mathbf{SP}_n$ is defined only for $t \in [t_n, 1)$, i.e., when all the n consumers are active.

In order to cope with the constraint $x_i = \max\{\sqrt{p\lambda c_i/(1-t)} - \lambda,\ 0\}$ in problem **P**, we consider dropping off the non-negative restrictions on advertisement expenditure x_i in **P**, which leads to a series of subproblems denoted by $\mathbf{D}_k$ defined as

$$\begin{aligned} &\max_{t\in[0,1)} f_M^k(t) = q\sum_{i=1}^{k} \alpha c_i \frac{x_i}{x_i+\lambda} - \alpha t \sum_{i=1}^{k} x_i, \\ &s.t. \qquad x_i = \sqrt{\frac{p\lambda c_i}{(1-t)}} - \lambda,\ i = 1,2,\ldots,k. \end{aligned} \tag{29}$$

Like Section 4.2, it is not difficult to check that $f_M^k(t)$ $(k = 1,2,\ldots,n)$ is a concave function of $t\,(t \in [0,1))$ under the assumption $q < 4p$. Similarly to inequality (18), we define the following condition:

$$\frac{2k\sqrt{\lambda}}{T_k} > \frac{(2p-q)}{\sqrt{p}}. \tag{30}$$

The optimal solution of the subproblem $\mathbf{D}_k$, denoted by $(\hat{t}^k, \hat{x}^k) \equiv \left(\hat{t}^k,\ (\hat{x}_i^k)_{i=1}^k\right)$, is characterized as follows, which can be seen as a corollary of Theorem 4.2.

Corollary 5.1. *Suppose $q < 4p$. If inequality (30) holds, then $\widehat{t}^k \in (0,1)$ is the unique solution of Equation (27); otherwise $\widehat{t}^k = 0$. Consequently, $\widehat{x}_i^k = \sqrt{\frac{p\lambda c_i}{(1-\widehat{t}^k)}} - \lambda,\ i = 1, 2, \ldots, k$.*

Recalling that we have assumed $\lambda > 0, \alpha > 0$, and $c_1 \geq c_2 \geq \ldots \geq c_n \geq 1$, the following corollary is obviously true according to Corollary 5.1.

Corollary 5.2. *Suppose $q < 4p$. For all $k = 1,\ 2, \ldots, n$ and $1 \leq i < j \leq k$, if $\widehat{x}_i^k \leq 0$ is the optimal solution for $\mathbf{D}_k$, then $\widehat{x}_j^k \leq 0$.*

Corollary 5.2 demonstrates that a consumer does not deserve any advertisement in $\mathbf{D}_k$ if another consumer with greater centrality receives none.

We now further explore the relationship between $(\widehat{t}^k, \widehat{x}^k)$s for different k.

Theorem 5.3. *Suppose $q < 4p$. For $1 \leq k < r \leq n$, if $\widehat{x}_i^r < 0$ for a consumer $i \leq k$, then $\widehat{x}_i^k < 0$.*

Proof. Since $\widehat{x}_i^k = \sqrt{\frac{p\lambda c_i}{(1-\widehat{t}^k)}} - \lambda$ is an increasing function of the participation rate $\widehat{t}^k$, we only need to prove that $\widehat{t}^k \leq \widehat{t}^r$.

If $\widehat{t}^k = 0$, then $\widehat{t}^k \leq \widehat{t}^r$ evidently holds. Hereafter, we assume $\widehat{t}^k > 0$, thus $\widehat{t}^k$ must be the unique stationary point of $f_M^k(t)$ according to Corollary 5.1. In other words, $\widehat{t}^k$ must satisfy Equation (27), i.e.,

$$2\left(\frac{k\sqrt{\lambda}}{T_k}\right)\left(1-\widehat{t}^k\right)^{\frac{3}{2}} - \frac{1}{\sqrt{p}}\left((q-p)\widehat{t}^k - (q-2p)\right) = 0. \tag{31}$$

To prove $\widehat{t}^k \leq \widehat{t}^r$, we only need to prove that $\frac{df_M^r(t)}{dt}|_{t=\widehat{t}^k} \geq 0$. In fact,

$$\begin{aligned}
&\frac{df_M^r(t)}{dt}\Big|_{t=\widehat{t}^k} \\
&= \frac{\alpha T_r\sqrt{\lambda}}{2}\left(1-\widehat{t}^k\right)^{-\frac{3}{2}}\left[2\left(\frac{r\sqrt{\lambda}}{T_r}\right)\left(1-\widehat{t}^k\right)^{\frac{3}{2}} - \frac{1}{\sqrt{p}}\left((q-p)\widehat{t}^k - (q-2p)\right)\right] \\
&\geq \frac{\alpha T_r\sqrt{\lambda}}{2}\left(1-\widehat{t}^k\right)^{-\frac{3}{2}}\left[2\left(\frac{k\sqrt{\lambda}}{T_k}\right)\left(1-\widehat{t}^k\right)^{\frac{3}{2}} - \frac{1}{\sqrt{p}}\left((q-p)\widehat{t}^k - (q-2p)\right)\right] \\
&= \frac{df_M^k(t)}{dt}\Big|_{t=\widehat{t}^k} \\
&= 0.
\end{aligned} \tag{32}$$

The inequality in (32) follows from $\frac{r}{T_r} \geq \frac{k}{T_k}$, which is easy to verify under our assumptions $c_1 \geq c_2 \geq \ldots \geq c_n \geq 1$ and $1 \leq k < r \leq n$. This completes the proof. □

Theorem 5.3 shows some sort of "monotonicity" of the optimization problem; an extension of social network will not worsen the value of a consumer. If a consumer receives no advertisement in a social network G, they also receive none in any sub-network of G.

For the subproblem $\mathbf{D}_k$ and its optimal solution $(\widehat{t}^k, \widehat{x}^k)$, if $\widehat{x}_i^k \geq 0$ for all i, we refer to $\mathbf{D}_k$ as having a "*potential solution*". The definition of the potential solution is equivalent to $\widehat{t}^k \in [t_k, 1)$. Theorem 5.4 below indicates that the optimal solution of the original problem $\mathbf{P}$ must be achieved at a potential solution for a subproblem $\mathbf{D}_k$.

Theorem 5.4. *Assume $q < 4p$. For problem $\mathbf{P}$ and its optimal solution (t^*, x^*), suppose k $(1 \leq k \leq n)$ is an integer such that $x_i^* > 0$ when $i \leq k$ and $x_i^* = 0$ when $i > k$. Then, the optimal solution of $\mathbf{D}_k$, i.e., $\left(\widehat{t}^k, \widehat{x}^k\right)$, satisfies $\widehat{t}^k = t^*$ and $\widehat{x}_i^k = x_i^*$ for all $i \leq k$.*

Proof. We only need to prove that $\widehat{t}^k = t^*$. We consider the cases $t^* > 0$ and $t^* = 0$ separately.

First, we consider the case when $t^* > 0$. In this situation, to prove $\widehat{t}^k = t^*$, we only need to prove that t^* is a stationary point of $f_M^k(t)$, *i.e.* $\frac{df_M^k(t)}{dt}|_{t=t^*} = 0$, due to the uniqueness of the stationary point of $f_M^k(t)$. As $x_k^* > 0$ and $x_{k+1}^* = 0$, we have $t_k < t^* \leq t_{k+1}$. If $\frac{df_M^k(t)}{dt}|_{t=t^*} \neq 0$, then three subcases require consideration:

(i) $\frac{df_M^k(t)}{dt}|_{t=t^*} < 0$. In this subcase, there exists an $\epsilon > 0$ such that $\epsilon < t^* - t_k$ and $f_M^k(t^*) < f_M^k(t^* - \epsilon)$. Noticing that $f_M^k(t) = \pi_M^n(t)$ when $t \in \left[t_k, t_{k+1}\right]$, we have $\pi_M^n(t^*) < \pi_M^n(t^* - \epsilon)$, which implies that (t^*, x^*) is not the optimal solution for problem $\mathbf{P}$.

(ii) $\frac{df_M^k(t)}{dt}|_{t=t^*} > 0$ and $t^* < t_{k+1}$. In this subcase, there exists an $\epsilon > 0$ such that $\epsilon < t_{k+1} - t^*$ and $f_M^k(t^*) < f_M^k(t^* + \epsilon)$. This is equivalent to $\pi_M^n(t^*) < \pi_M^n(t^* + \epsilon)$, which indicates that (t^*, x^*) is not the optimal solution for problem $\mathbf{P}$.

(iii) $\frac{df_M^k(t)}{dt}|_{t=t^*} > 0$ and $t^* = t_{k+1}$. In this subcase, $k < n$ always holds; otherwise, $t^{n*} = t_{n+1} = 1$, which is impossible because a participation rate of 1 will inevitably lead to infinite expenditures on all consumers and a negative profit for the manufacturer (i.e., $\pi_M^n = -\infty$).

Define

$$t_{k+1}^+ = \begin{cases} \min\left\{t_s | s > k+1,\ t_s \neq t_{k+1}\right\}, \text{if } t_{k+1} \neq t_n, \\ 1, \qquad\qquad\qquad\qquad\qquad\quad \text{if } t_{k+1} = t_n, \end{cases} \tag{33}$$

then $t_{k+1}^+ > t_{k+1} = t^*$. Using the same method as in the proof of Theorem 5.3, we have $\frac{df_M^{k+1}(t)}{dt}|_{t=t^*} > \frac{df_M^k(t)}{dt}|_{t=t^*} > 0$. Therefore, there exists $\epsilon > 0$ such that $\epsilon < t_{k+1}^+ - t^*$ and $f_M^k(t^*) = f_M^{k+1}(t^*) < f_M^{k+1}(t^* + \epsilon)$. Consequently, we have $\pi_M^n(t^*) = \pi_M^k(t^*) = f_M^k(t^*) < f_M^{k+1}(t^* + \epsilon) = \pi_M^{k+1}(t^* + \epsilon) = \pi_M^n(t^* + \epsilon)$, indicating that (t^*, x^*) is not the optimal solution of $\mathbf{P}$.

Summarizing these three subcases, we conclude that $\frac{df_M^k(t)}{dt}|_{t=t^*} = 0$, i.e., $\widehat{t}^k = t^*$.

Now we consider the case $t^* = 0$ and prove that $\widehat{t}^k = 0$. Otherwise, if $\widehat{t}^k > 0$, then we have $f_M^k\left(\widehat{t}^k\right) > f_M^k(0)$. Since $q < 4p$, $f_M^k(t)$ is a concave function of t for $t \in [0, 1)$, meaning that $\frac{df_M^k(t)}{dt}|_{t=t^*} > 0$. Using the same method as in the proofs for subcases (ii) and (iii) for $t^* > 0$, it is not difficult to demonstrate that $(0, x^*)$ is not the optimal solution for $\mathbf{P}$, which leads to a contradiction. Therefore, we must have $\widehat{t}^k = 0 = t^*$. This completes the proof. □

Theorem 5.4 delineates that the optimal solution of $\mathbf{P}$ must correspond to a potential solution of $\mathbf{D_k}$ that satisfies $t^* \in [t_k, t_{k+1}]$. Leveraging these theorems collectively, it is feasible to develop an algorithm to ascertain a solution for the primary problem $\mathbf{P}$ by addressing all $\mathbf{D_k}$s with associated potential solutions.

Algorithm 2. Step 0. Relabel the consumers as $c_1 \geq c_2 \geq \ldots \geq c_n$. Compute $t_i = 1 - pc_i/\lambda$ $(i = 1, 2, \ldots, n)$, $t_{n+1} = 1$, and $K = \min\{1 \leq i \leq n+1 | t_i \geq 0\}$. Assign $t^* = 0, \pi_M^* = 0$, and set $L = n$.

Step 1. Solve $\mathbf{D}_L$ and identify its optimal participation rate $\widehat{t}^L$ (according to Corollary 5.1). If $t_L < \widehat{t}^L \leq t_{L+1}$, proceed to Step 2; if $\widehat{t}^L > t_{L+1}$, proceed to Step 3; otherwise, proceed to Step 4.

Step 2. Compute $\left(\widehat{t}^L, \widehat{x}^L\right)$ as a potential solution of $\mathbf{D}_L$ and the manufacturer's expected profit $\pi_M^L\left(\widehat{t}^L\right)$; if $\pi_M^L\left(\widehat{t}^L\right) > \pi_M^*$, set $t^* = \widehat{t}^L, \pi_M^* = \pi_M^L\left(\widehat{t}^L\right)$.

Step 3. If $c_L = c_1$, set $L = -1$, otherwise set $L = \max\{1 \leq i \leq L-1 | c_i > c_L\}$. Proceed to Step 5.

Step 4. $\left(\widehat{t}^L, \widehat{x}^L\right)$ is not a potential solution. If $\widehat{t}^L \leq t_1$, set $L = -1$; otherwise, set $L = \max\left\{1 \leq i \leq L-1 | \widehat{t}^L > t_i\right\}$ and proceed to Step 1.

Step 5. If $L < K - 1$, output t^* and π_M^*, and stop; otherwise, proceed to Step 1.

In Algorithm 2, L is utilized to track the range of $\mathbf{D_k}$s that have been examined, implying that all $k > L$ have already been processed in the algorithm.

Step 1 solves the current problem $\mathbf{D_L}$ and considers three distinct scenarios:

If $t_L < \widehat{t}^L \leq t_{L+1}$, $\mathbf{D_L}$ has a potential solution. Step 2 calculates the manufacturer's expected profit and compares it with the current best to decide whether to update it. The algorithm then proceeds to Step 3 to update L by resetting it to the next i with a greater centrality.

If $\widehat{t}^L > t_{L+1}$, $\mathbf{D_L}$ has a potential solution, but the solution was already examined previously. According to Theorem 5.4, it will not be the solution for $\mathbf{P}$, so the algorithm bypasses the calculation of expected profits, and only updates L in Step 3.

If $\widehat{t}^L \leq t_L$, $\mathbf{D_L}$ has no potential solution. Moreover, according to Theorem 5.3, for any consumer i for which $\widehat{t}^L \leq t_i$, $\mathbf{D_i}$ will lack a potential solution due to $\widehat{x}_i^i < 0$. Consequently, the algorithm employs Step 4 to bypass these consumers and continue from $L = \max\left\{1 \leq i \leq L-1 | \widehat{t}^L > t_i\right\}$.

As in Algorithm 1, Step 5 ensures that the algorithm terminates when $t_L \leq 0$.

Algorithm 2 investigates all $\mathbf{D_k}$s with potential solutions. According to Theorem 5.4, the optimal solution of $\mathbf{P}$ must correspond to a potential solution for a certain $\mathbf{D_k}$, so we have the following theorem.

Theorem 5.5. *Suppose $q < 4p$. The optimal participation rate t^* of M, along with the optimal expected profit π_M^*, can be acquired using Algorithm 2.*

5.2. **An example of Algorithm 2.** We use a numerical example to illustrate the detailed procedures of Algorithm 2. Assume $\alpha = 0.5, p = q = 1, \lambda = 2.5, n = 10$, and a social network with a structure as depicted in Figure 1. The corresponding communication matrix is

$$E = \begin{pmatrix} 0 & 0.25 & 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0.25 & 0 \\ 0.5 & 0 & 0 & 0 & 0.25 & 0.125 & 0.125 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

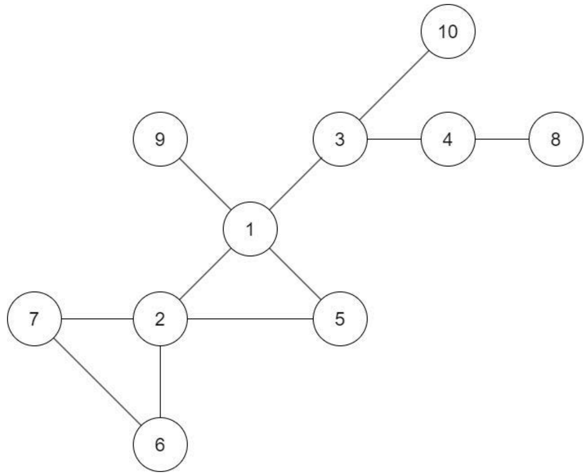

FIGURE 1. An example of a social network with 10 consumers.

The centrality vector $c = \mathbf{1}^T \left[I - (1-\alpha)E\right]^{-1} = (3.4775, 2.6535, 2.6206, 2.0887, 1.7664, 1.5545, 1.5545, 1.5222, 1.4347, 1.3276)$. It is easy to see that the consumers are labeled to satisfy $c_1 \geq c_2 \geq \ldots \geq c_n$.

Now we show how Algorithm 2 is exhibited for this example. We divide the process into iterations to clarify the procedures. Each iteration signifies a return to Step 1, requiring the solution of a new problem $\mathbf{D_k}$.

Step 0. Compute $(t_k)_{k=1}^n = (-0.3910, -0.0614, -0.0483, 0.1645, 0.2935, 0.3782, 0.3782, 0.3911, 0.4261, 0.4690)$. Therefore, $K = 4$. Set $t^* = 0, \pi_M^* = 0$, and $L = n = 10$.

Iteration 1. Solve $\mathbf{D_{10}}$. Since $\widehat{t}^{10} = 0.4201 < 0.4690 = t_{10}$, $\left(\widehat{t}^{10}, \widehat{x}^{10}\right)$ is not a potential solution.

Proceed to Step 4. Set $L = \max\left\{1 \leq i \leq L-1 | \widehat{t}^L > t_i\right\} = 8$, and then move to Step 5.

$L = 8 > K - 1 = 3$, so proceed to Step 1.

Iteration 2. Solve $\mathbf{D_8}$. Given $\widehat{t}^8 = 0.4049 > t_8$, proceed to Step 2.

Since $\widehat{t}^8 = 0.4049 < 0.4261 = t_9$, so $\left(\widehat{t}^8, \widehat{x}^8\right)$ is a potential solution. Compute $\widehat{x}^8 = (1.3221, 0.8387, 0.8180, 0.4621, 0.2240, 0.0554, 0.0554, 0.028)$.

Since $\pi_M^L\left(\widehat{t}^L\right) = 0.7655 > \pi_M^* = 0$, set $t^* = \widehat{t}^8 = 0.4049, \pi_M^* = \pi_M^L\left(\widehat{t}^L\right) = 0.7655$.

Since $c_8 \neq c_1$, set $L = \max\{i | c_i \neq c_8\} = 7$ and proceed to Step 5.

$L = 7 > K - 1 = 3$, so proceed to Step 1.

Iteration 3. Solve $\mathbf{D_7}$. Since $\widehat{t}^7 = 0.3964 > t_8 = 0.3911$, the solution of $\mathbf{D_7}$ has already been considered. Proceed to Step 3.

Since $c_7 \neq c_1$, set $L = \max\{i | c_i \neq c_7\} = 5$ and proceed to Step 5.

$L = 5 > K - 1 = 3$, so proceed to Step 1.

Iteration 4. Solve $\mathbf{D_5}$. Since $\widehat{t}^5 = 0.3710 > t_5$ and $\widehat{t}^5 = 0.3710 < 0.3782 = t_6$, $\left(\widehat{t}^5, \widehat{x}^5\right)$ is a potential solution. Proceed to Step 2.

Compute $\widehat{x}^5 = (1.2178, 0.7476, 0.7275, 03813, 0.1497)$.

Since $\pi_M^L\left(\widehat{t}^L\right) = 0.7603 < \pi_M^* = 0.7655$, the algorithm does not update t^* and π_M^*.

Since $c_5 \neq c_1$, set $L = \max\{i|c_i \neq c_5\} = 4$ and proceed to Step 5.

$L = 4 > K - 1 = 3$, so proceed to Step 1.

Iteration 5. Solve $\mathbf{D_4}$. Since $\widehat{t}^4 = 0.3546 > t_5 = 0.2935$, the solution of $\mathbf{D_4}$ has been considered before. Proceed to Step 3.

Since $c_4 \neq c_1$, set $L = \max\{i|c_i \neq c_4\} = 3$, and proceed to Step 5.

$L = 3 = K - 1 = 3$, so proceed to Step 1.

Iteration 6. Solve $\mathbf{D_3}$. Since $\widehat{t}^3 = 0.3378 > t_4 = 0.1645$, the solution of $\mathbf{D_3}$ has been examined before. Proceed to Step 3.

Since $c_3 \neq c_1$, set $L = \max\{i|c_i \neq c_3\} = 2$, and proceed to Step 5.

As $L = 2 < K - 1 = 3$, the algorithm stops, and outputs $t^* = \widehat{t}^8 = 0.4049$ and

$$(x_i^*)_{i=1}^8 = \widehat{x}^8 = (1.3221, 0.8387, 0.8180, 0.4621, 0.2240, 0.0554, 0.0554, 0.0287)$$

with $x_9^* = x_{10}^* = 0$ as the optimal solution. The corresponding expected profit of M is $\pi_M^* = 0.7655$.

We can also calculate that the total advertisement expenditure is 3.8044, the expected profit of R is $\pi_R^* = 0.4036$, and the total expected profit of M and R is $\pi_M^* + \pi_R^* = 1.1691$.

This example provides a comprehensive demonstration of the different scenarios one might encounter when employing Algorithm 2. As is shown, Algorithm 2 demonstrates efficiency by using only 6 iterations to pinpoint the optimal solution, in contrast to the 8 subproblems necessitated by Algorithm 1. Furthermore, the algorithm only calculates the manufacturer's expected profit twice, which further emphasizes its efficiency.

5.3. **Advantage of Algorithm 2.** We can now discuss Algorithm 2's efficiency compared to its benchmark Algorithm 1. Both algorithms use the same approach to solve the participation rate (t) in a subproblem; therefore, their efficiency can be evaluated based on two key criteria: N_s, the number of subproblems to solve, and N_p, the number of expected profits that need to be calculated.

In Algorithm 1, $N_s = N_p = n - K + 2$. This implies that $(n - K + 2)$ subproblems must be solved, and the corresponding $(n - K + 2)$ expected profits must be calculated for the manufacturer.

In Algorithm 2, $N_p \leq N_s \leq n - K + 2$. It solves no more than $(n - K + 2)$ subproblems. Moreover, Steps 3 and 4 introduce opportunities to skip certain subproblems, reducing N_s. Specifically, $N_s = n - K + 2$ only in the following scenario: Each time when searching for $L = \max\{1 \leq i \leq L - 1|c_i > c_L\}$ during Step 3, or when searching for $L = \max\left\{1 \leq i \leq L - 1|\hat{t}^L > t_i\right\}$ during Step 4, $(L - 1)$ is the result, indicating that no subproblems can be skipped. Otherwise, $N_s < n - K + 2$, showing improved efficiency.

Regarding N_p, Algorithm 2 only calculates the expected profit for the manufacturer when $t_L < \hat{t}^L \leq t_{L+1}$, so $N_p \leq N_s$. $N_p = N_s$ occurs only when $t_L < \hat{t}^L \leq t_{L+1}$ each time we skip to Step 2. Otherwise, $N_p < N_s$, indicating further efficiency improvements.

Overall, Algorithm 2 is consistently more efficient than Algorithm 1, showing no performance degradation even in the worst case.

5.4. **Algorithm performance under specific topology structures of social networks.** Beyond the demonstrated example where $n = 10$ in Section 5.2, Algorithm 2 exhibits good scalability as the social network expands to include more consumers, with both N_s and N_p having a maximum value of $(n - K + 2)$. The efficiency of the algorithm is influenced by the topology of the social network. Generally, Algorithm 2 performs more efficiently when many consumers have similar or closely matched centralities. The following examples illustrate three special types of social networks in which Algorithm 2 effectively reduces computational load while maintaining the optimality of the solution.

5.4.1. *Complete network.* In a complete social network, all consumers can communicate with all others. That is, $e_{ij} = \frac{1}{n-1}$, $\forall i \neq j$. Figure 2 shows a complete network with $n = 4$.

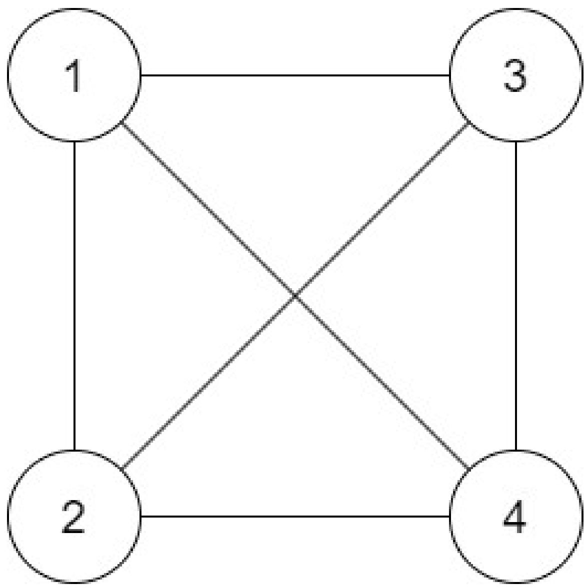

FIGURE 2. A complete social network with 4 consumers.

In a complete network, all centralities are identical, so Algorithm 2 always has $N_s = N_p = 1$, regardless of the value of n.

5.4.2. *Star network.* In a star network, there is a key opinion leader in the center of the network, and all other consumers only communicate with them. That is, $e_{i1} = 1, e_{1i} = \frac{1}{n-1}$, $\forall i \neq 1$; $e_{ij} = 0$, for other (i, j). Figure 3 shows a star network with $n = 5$.

In a star network, all centralities are identical except for the center node, so Algorithm 2 always has $N_p \leq N_s \leq 2$.

5.4.3. *m-center network.* An m-center network is a social network consisting of $m\,(1 \leq m \leq n)$ key opinion leaders, each of whom communicates with the other opinion leaders and has a set of followers who only communicate with them. The numbers of followers for each opinion leader are denoted as $\{a_1, \ldots, a_m\}$. Let η $(0 < \eta < 1)$ represent the proportion of communication among the opinion leaders, and define $b_0 = m$ and $b_k = m + \sum_{i=1}^{k} a_i\,(1 \leq k \leq m)$. An m-center network can then be expressed as follows: $e_{ij} = \frac{\eta}{m-1}$, $i \neq j \leq m$; $e_{ij} = \frac{1-\eta}{a_i}, i \leq m, b_{i-1} < j \leq b_i$; $e_{ji} = 1, i \leq m, b_{i-1} < j \leq b_i$; $e_{ij} = 0$, for other (i, j). Figure 4 shows a 3-center

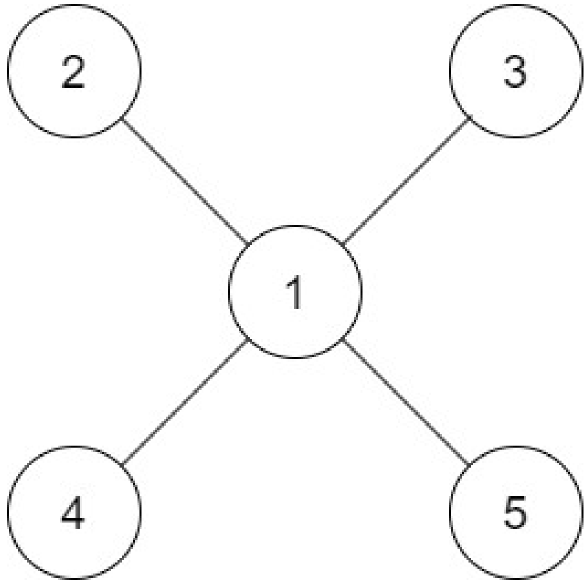

FIGURE 3. A star social network with 5 consumers.

social network where $n = 9, m = 3, a_1 = 3, a_2 = 2,$ and $a_3 = 1$ (the parameter η can take any value between 0 and 1).

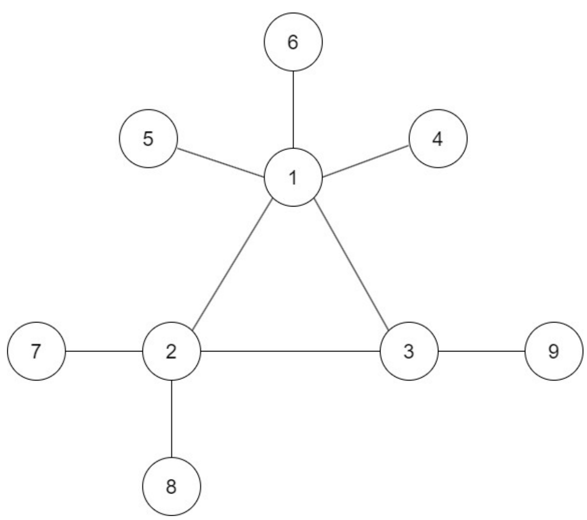

FIGURE 4. A 3-center social network with 9 consumers.

The m-center networks reflect characteristics of real-world social networks, where a few opinion leaders exert the most influence, disseminating information to other consumers. Notably, for each opinion leader, the centralities of their followers are identical. Therefore, for an m-center network, the complexity of Algorithm 2 satisfies $N_p \leq N_s \leq 2m$, regardless of the number of followers.

The three examples above illustrate how Algorithm 2 enhances efficiency. The algorithm performs particularly well when Steps 3 and 4 are able to skip a large number of consumers simultaneously. A typical scenario occurs when multiple consumers exhibit similar or closely matched centralities. In such cases, these consumers collectively require solving a subproblem at most once, regardless of their actual number. This substantially reduces both N_s and N_p, thereby improving the overall efficiency of the algorithm.

6. The influence of model parameters on strategies and profits.

Subsection 5.2 presents a numerical example illustrating the determination of the optimal participation rate. Building on the same social network, this section examines the effects of model parameters on the optimal strategies and profits of M and R.

6.1. The influence of α and λ.

This subsection focuses on the influence of advertisement sensitivity α and advertisement resistance λ. Two main questions are addressed: 1) How variations in α and λ influence the participation rate t, and 2) how they affect the profits of M and R. To isolate the effects of α and λ, we fix $p = q = 1$. The impact of p and q will be discussed in Subsection 6.2.

Figure 5 illustrates how variations in α and λ influence the participation rate (t). The results indicate that t generally increases with higher α, and also tends to rise as λ increases. However, this trend is not strictly monotonic. In certain ranges, an increase in λ leads to a decline in t. For instance, when $\alpha = 0.35$, t decreases from 0.3786 to 0.3553 as λ increases from 3 to 3.5, before rising again to 0.3833 as λ continues to increase to 4. In more extreme cases, such as when $\alpha = 0.8$ and $\lambda = 3.5$, t abruptly drops to zero.

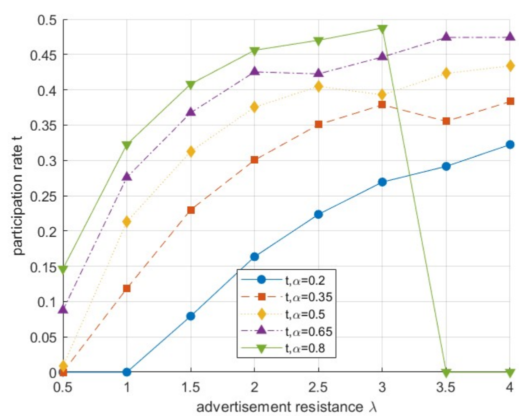

FIGURE 5. The influence of α and λ on participation rate t.

Changes in participation rate directly affect the profits of both M and R. Figures 6 and 7 show how α and λ influence profits, with trends more pronounced than those for t. Both M and R experience declining profits as either α or λ increases.

The effects of α and λ can be explained by their role as negative factors in advertising outcomes.

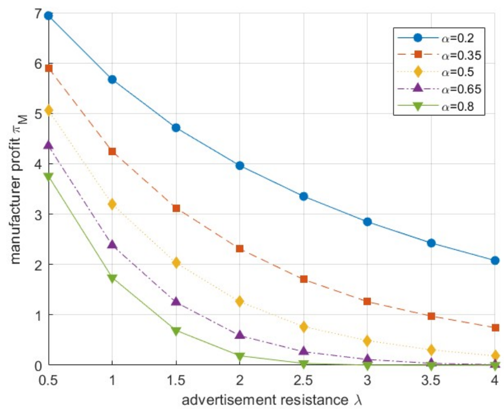

FIGURE 6. The influence of α and λ on manufacturer's profit.

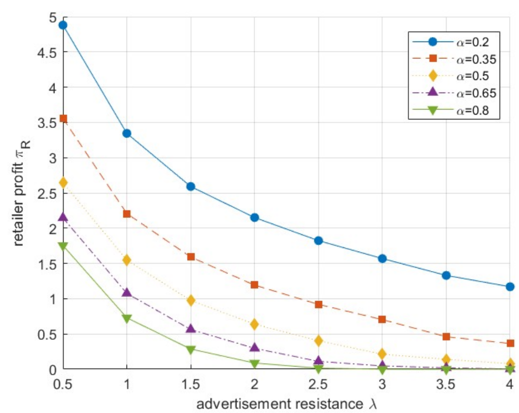

FIGURE 7. The influence of α and λ on retailer's profit.

The influence of λ is straightforward: Higher advertisement resistance offsets the returns on advertising expenditure. The detrimental effect of advertising sensitivity (α) arises from the binary demand of consumers and the cost-per-click (CPC) advertising scheme. In the CPC model, α also serves as a cost factor. A higher α indicates that consumers rely more on advertisements for information and click on them more frequently, resulting in higher costs for the advertiser. However, due to the binary nature of consumer demand, increased exposure to advertisements does not expand potential market demand, but merely reallocates it. Consequently, the higher advertising costs associated with a larger α are not offset by a proportional increase in demand. As a result, higher advertising sensitivity (α) diminishes the ability to leverage the social network, increases CPC costs, and reduces the efficiency of the advertising budget.

When α and λ are low, advertising faces little resistance in attracting consumers. Even a modest advertising investment can substantially increase total demand, motivating the retailer to advertise actively within the social network. In this case, the manufacturer can maintain a low participation rate while still earning high profits, as shown in the bottom left of Figure 5 and the top left of Figure 6. The retailer, benefiting from strong advertising effects, also enjoys high profits even without significant advertising subsides from the manufacturer, as illustrated in the top left of Figure 7.

As α and λ increase, resistance to advertising becomes more significant. Retailer-driven advertising alone no longer generates sufficient profit to sustain advertising efforts. Because the manufacturer's sales, and thus the profits, depend on the retailer's advertising, the manufacturer needs to subsidize a larger share of the retailer's advertising expenses to maintain or increase the total advertising budget. This dynamic leads to a rapid rise in t, as shown in the middle region of Figure 5. However, since the increase in t is a response to deteriorating market conditions, profits for both M and R still decline, as shown in Figures 6 and 7.

When both α and λ become sufficiently high, advertising can become unprofitable despite a high participation rate, prompting the retailer to withdraw from certain market segments. As a result, the increasing trend in the participation rate t is interrupted, and a decline in t may occur as λ rises. For instance, when $\alpha = 0.35$ and λ increases from 3 to 3.5, the retailer is forced to abandon a significant portion of the market, advertising to only 5 consumers at $\lambda = 3.5$, compared to 9 consumers when $\lambda = 3$. After retreating from consumers with low centrality, the retailer focuses on more profitable consumers, which restores the profitability of advertising, requiring a smaller participation rate to sustain the same advertising impact in a reduced market. Consequently, the participation rate declines.

For higher α values (e.g., $\alpha = 0.65$), this "shrinkage effect" may occur when λ exceeds 2. In contrast, for lower α values (e.g., $\alpha = 0.2$), the retailer maintains a broader advertising reach even when λ reaches 4, causing the participation rate to remain in a monotonically increasing trend.

6.2. **The influence of q and p.** In this subsection, we focus on the effects of gross profit margins earned by M and R, i.e., q and p. We examine how they affect the participation rate and the profits of M and R. To isolate these effects, we fix $\alpha = 0.5$ and $\lambda = 2.5$.

Figure 8 illustrates how q and p affect the participation rate t. Note that some data points have been removed to satisfy the constraint $q < 4p$. The influence of the manufacturer's profit margin q on t is straightforward: Increases in q consistently lead to higher participation rates. This is because a higher q means the manufacturer earns more from each product sold, and is therefore more willing to subsidize the retailer's advertising efforts.

The effect of the retailer's profit margin p on t is more nuanced. As shown in Figure 8, in most cases t decreases as p increases; however, when q is low (e.g., $q = 0.2$), t initially rises from zero to a positive value before beginning a steady decline. This occurs because, when both p and q are low, sales revenues may not cover advertising costs, prompting both M and R to withdraw from the market, resulting in $t = 0$. Once a threshold of "active margin" is reached, as p continues to rise, R becomes more sufficiently motivated to advertise even without substantial subsidies from M, leading to a decline in t.

Figure 9 shows how p and q affect the profit of M. The profit rises as q increases, a straightforward outcome as earning more from each sale is inherently beneficial. The profit also increases with p because a higher retailer profit margin stimulates

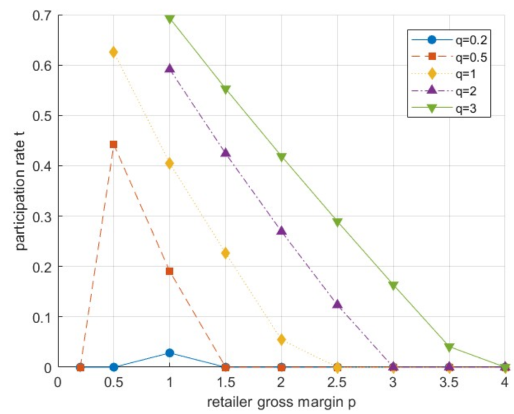

FIGURE 8. The influence of q and p on participation rate t.

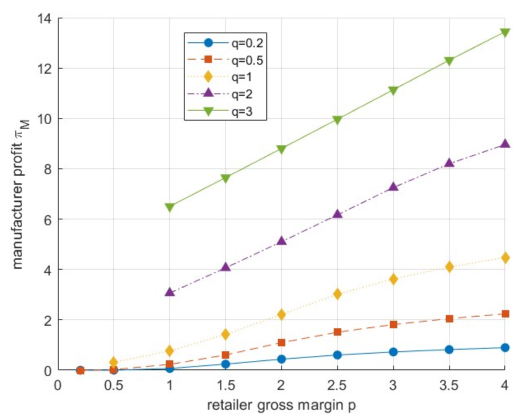

FIGURE 9. The influence of q and p on the manufacturer's profit.

greater advertising activity and allows M to maintain a lower participation rate, thereby reducing subsidy costs while benefiting from increased sales.

Figure 10 illustrates the retailer's profit response to changes in p and q. As p increases, the profit of R rises due to higher per-unit earnings and the ability to invest more in advertising. The profit of R also tends to increase with q because higher q raises the participation rate t, enabling R to expand advertising with the same budget, which boosts sales and profits. Notably, at the right end of Figure 10, the profits of R under different q values converge. This convergence occurs because p becomes sufficiently large relative to q and advertising costs, allowing R to sustain optimal advertising levels independently without requiring a cooperative advertising scheme (i.e., $t = 0$).

7. **Profitability of cooperative advertising.** The example in Subsection 5.2 also demonstrates how cooperative advertising improves overall outcomes. In the example, cooperative advertising results in a total advertising expenditure of 3.8044,

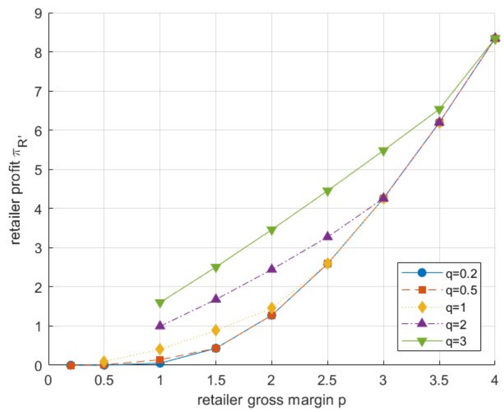

FIGURE 10. The influence of q and p on the retailer's profit.

with the profits of M and R being 0.7655 and 0.4036, respectively. In contrast, it is not difficult to show that without cooperative advertising ($t = 0$), the total advertising expenditure will be only 0.5837, and the profits of M and R are 0.3339 and 0.0421, respectively. Compared with the non-cooperative advertising scenario, the retailer's expected profit increases by 858.67%, while the manufacturer's profit increases by 129.26%. In this section, we further examine whether the substantial profit enhancement from cooperative advertising is common under different market conditions and identify the key factors that influence its effectiveness.

7.1. **Cooperative advertising profitability under different α and λ.** In this subsection, we vary the parameters α and λ to examine whether the substantial profit-enhancing effect persists under different conditions. To isolate the effects of α and λ, we fix $p = q = 1$. The profitability under different p and q will be discussed in Section 7.2.

Figures 11 and 12 show that cooperative advertising increases the expected profits for both M and R. Regardless of the values of α and λ, profits are higher when cooperative advertising is implemented. The effect is more pronounced for R than for M because R bears the direct cost of advertising. The profit gain is especially significant when α and λ are high.

Figures 13 and 14 plot the total advertising expenditure and the number of active consumers (i.e., those receiving non-negative advertising expenditure), which provide insight into how cooperative advertising enhances sales and the profits of M and R. By sharing part of the advertising expenditure of R, cooperative advertising strongly incentivizes R to expand the total advertising investment, extend advertising coverage, and target consumers who would otherwise be unprofitable without subsidies. These efforts result in higher expected profits. As shown in Figure 13, the total advertising expenditure is substantially greater than that of the case without cooperative advertising.

Referring to Subsection 5.1, the increases in total advertising expenditure and profits are closely related to the participation rate. On the left side of Figure 13, where the advertisement resistance is low, the participation rate remains low, and cooperative advertising has only a modest effect. In contrast, on the right side of

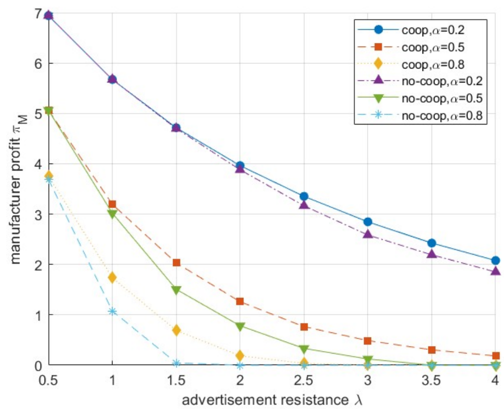

FIGURE 11. Profitability of co-op ad for M under different α and λ.

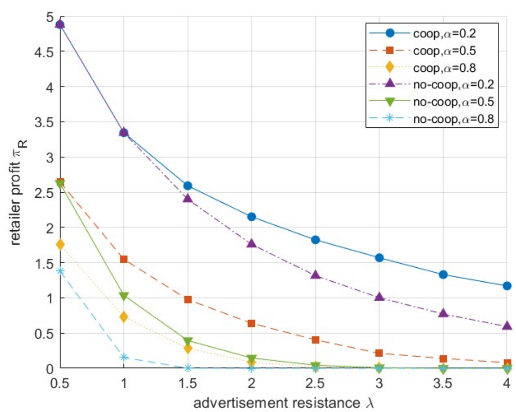

FIGURE 12. Profitability of co-op ad for R under different α and λ.

Figure 13, where the advertisement resistance is high, R cannot cover the advertising costs solely from sales revenue. Here, a high participation rate is essential to maintaining a substantial total advertising investment. Figure 14 shows that cooperative advertising also increases the number of consumers R targets. For example, when $\alpha = 0.8$, R exits the market once λ exceeds 2 in the absence of cooperative advertising, but continues to advertise to 8 out of 10 consumers when cooperative advertising is in place.

7.2. **Cooperative advertising profitability under different q and p.** In this subsection, we examine the profitability of cooperative advertising under different gross profit margins of M and R. To isolate the effect of these margins, we fix $\alpha = 0.5$ and $\lambda = 2.5$.

As shown in Figures 15 and 16, cooperative advertising increases the profits of M and R until higher values of p cause the participation rate to fall to zero. The

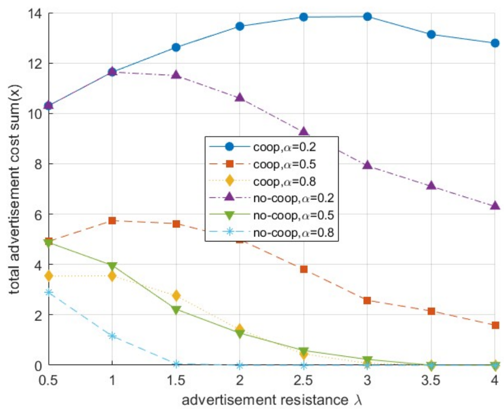

FIGURE 13. Total advertising expenditure under different α and λ.

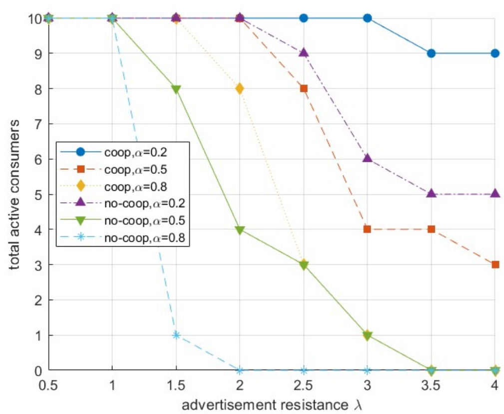

FIGURE 14. Total active consumers under different α and λ.

profit gains from cooperative advertising are more pronounced when p and q are relatively low.

The underlying mechanism of cooperative advertising is consistent across cases. As illustrated in Figures 17 and 18, although cooperative advertising is less effective in expanding the number of consumers reached by R, it still enhances profits primarily by substantially increasing total advertising expenditure. When p and q are relatively low, cooperative advertising plays a critical role in sustaining a sizable advertising investment, which results in a higher t and a greater relative contribution of cooperative advertising. Conversely, when p becomes sufficiently high, cooperative advertising is no longer necessary; consequently, the curves for total advertising expenditure and profit converge.

7.3. **Cooperative advertising profitability in different social network structures.** This subsection examines how the structure of the social network influences

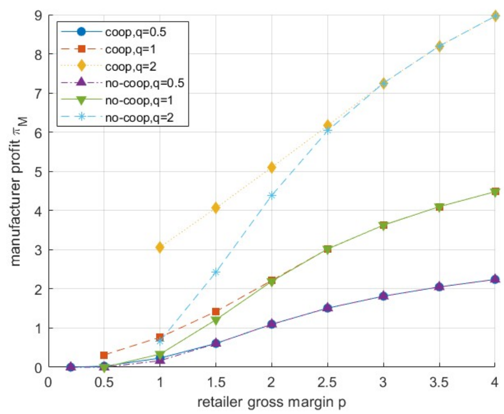

FIGURE 15. Profitability of co-op ad for M under different q and p.

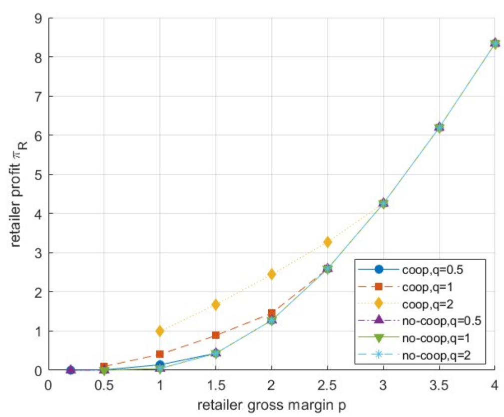

FIGURE 16. Profitability of co-op ad for R under different q and p.

the participation rate, the advertising expenditure, and the profits of M and R. In specific, we focus on the role of opinion leaders.

As noted in Subsection 5.2, the consumers connected to the largest number of others, namely, consumers 1, 2, and 3, exhibit the highest centralities. In this subsection, we define the top τ consumers with the highest centralities as the opinion leaders in the social network. The number of opinion leaders, τ, can be chosen subjectively based on the distribution of centralities in the network. To capture their influence, we introduce a parameter referred to as the leadership index. In a communication matrix where consumers are relabeled to satisfy $c_1 \geq c_2 \geq \cdots \geq c_n$, as in the communication matrix E in Equation (34), if we select τ opinion leaders and set a value for ρ, we multiply the first τ columns of the communication matrix E by ρ, and normalize the matrix by row to satisfy $\sum_j e_{ij} = 1$ for all i. For example, if we choose $\tau = 3$, then when $\rho = 1$, the matrix E remains unchanged, whereas

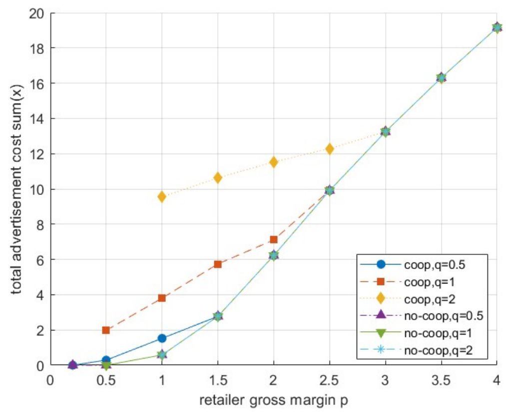

FIGURE 17. Total advertising expenditure under different q and p.

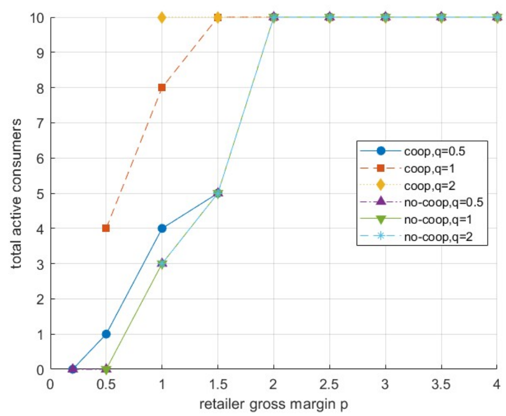

FIGURE 18. Total active consumers under different q and p.

when $\rho = 3$, the matrix is modified as follows:

$$E = \begin{pmatrix} 0 & 0.375 & 0.375 & 0 & 0.125 & 0 & 0 & 0 & 0.125 & 0 \\ 0.75 & 0 & 0 & 0 & 0.125 & 0.0625 & 0.0625 & 0 & 0 & 0 \\ 0.75 & 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0.125 \\ 0 & 0 & 0.75 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We analyze the cooperative advertising profitability in social networks under different numbers of opinion leaders ($\tau = 1, 2, 3$) and varying leadership indices ($0.25 \leq \rho \leq 3$). In this subsection, we fix $\alpha = 0.5, \lambda = 2.5,$ and $p = q = 1$.

As an immediate observation, the advantage of cooperative advertising remains consistent across different numbers of opinion leaders and varying leadership indices ρ. As shown in Figures 19 and 20, the profits of both M and R increase significantly under cooperative advertising. Even in cases where cooperative advertising is least effective, such as the three-leader, $\rho = 3$ scenario, the profit of M is still 50% higher with cooperative advertising than that without it, while the profit of R is approximately 3 times as high. Moreover, the higher the value of ρ, the greater the profits of both M and R. This is because a higher ρ concentrates consumers' centrality, enabling R to target advertisements more effectively.

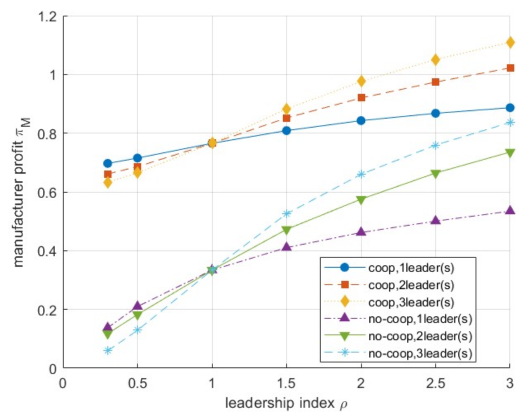

FIGURE 19. Profitability of co-op ad for M under different τ and ρ.

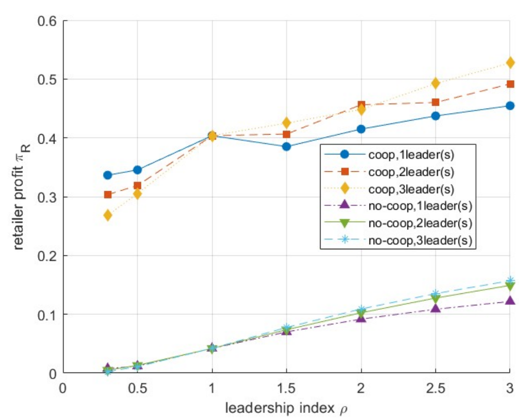

FIGURE 20. Profitability of co-op ad for R under different τ and ρ.

The superior performance of cooperative advertising is primarily due to greater advertising investment and the larger market share it subsequently captures. As shown in Figures 21 and 22, both the total advertising expenditure and the number of active consumers are greater under cooperative advertising.

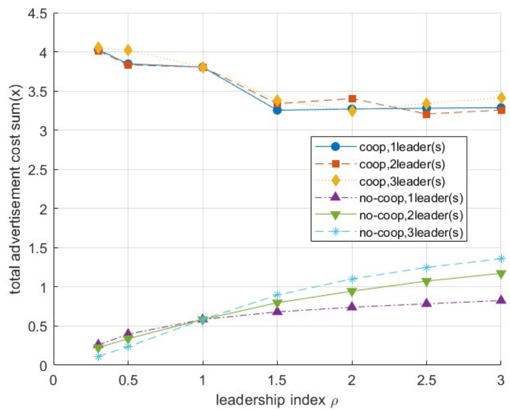

FIGURE 21. Total advertising expenditure under different τ and ρ.

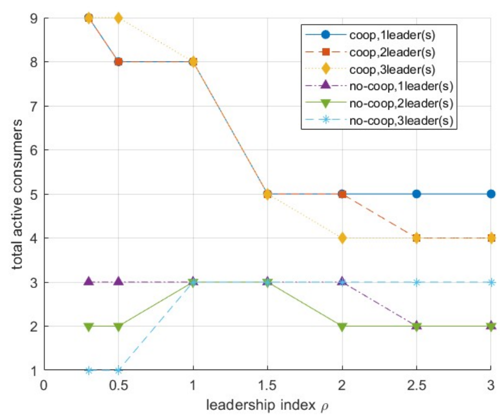

FIGURE 22. Total active consumers under different τ and ρ.

An additional phenomenon is that total advertising expenditure increases with ρ in the absence of cooperative advertising, but decreases when cooperative advertising is present. This occurs because a higher leadership index reshapes the distribution of centrality: Opinion leaders gain more centrality, while ordinary consumers lose importance. Without cooperative advertising, where the retailer alone struggles to afford substantial advertising, the concentration of value encourages the retailer to allocate more resources to opinion leaders, thereby raising total expenditure. However, under cooperative advertising, the overall advertising level is already relatively high, so the diminished importance of ordinary consumers leads to a reduction in total costs.

We also observe that total sales tend to increase as the number of opinion leaders rises. This is because greater centrality concentration to opinion leaders allows the retailer to capture a larger share of the market by targeting these leaders. On the other hand, this pattern appears less pronounced when $\rho < 1$, since the influence of opinion leaders weakens and their role may be replaced by other consumers.

8. **Conclusion.** This paper adopts the voter model proposed in [5] to analyze the cooperative advertising scheme within a social network. In a network where consumers purchase a single indivisible product and receive information from both advertisements and adjacent consumers, we derive a new demand function, calculate the expected profits for both the manufacturer and retailer, and identify the optimal strategies for both parties by solving the Stackelberg game.

This work makes several primary contributions in its mathematical derivations. First, we obtain closed-form expressions for the profits of both the manufacturer and the retailer in a social network setting. Second, exploiting structural properties of the optimization problem, such as convexity, we design a baseline algorithm that identifies the manufacturer's optimal participation rate by solving all subproblems. Third, leveraging the concept of potential solutions, we develop a more efficient algorithm. A potential solution is one in which the optimal participation rate yields positive advertising expenditures for all selected consumers. We prove that the Stackelberg game's optimal solution must be a potential solution of one of its surrogate problems, and we design an improved algorithm by enumerating all potential solutions across a sequence of surrogate problems. We establish the correctness of the improved algorithm, demonstrate its scalability across diverse social network sizes and topologies, and explain how it improves efficiency relative to the baseline. We also present several representative social network structures in which the improved algorithm delivers substantial speedups, illustrating how network topology influences algorithmic performance.

Several key managerial insights emerge from this work. First, we propose a novel cooperative advertising model. Unlike utility-based approaches, our model has distinctive features that make it well-suited to specific markets, such as consumer electronics and automobiles. Second, we conduct a series of numerical experiments to examine how model parameters influence the optimal participation rate and the profits of both parties. Finally, using these numerical examples, we assess the profitability of cooperative advertising under a variety of conditions. Across different advertisement parameters, gross profit margins, and leadership indices, we find that cooperative advertising significantly increases both parties' profits in most scenarios. These findings provide practical guidance for participants in cooperative advertising, highlighting three major factors for strategy development: The structure of the social network, market sensitivity and resistance to advertising, and the distribution of gross profit margins.

- The structure of the social network determines the pool of potential advertising targets. When regular consumers exhibit strong mutual communication, the network may include many profitable targets beyond opinion leaders. In such cases, the manufacturer should consider raising the participation rate to expand the advertising scope and overall budget, a proactive strategy that tends to increase profits for both parties.
- Market sensitivity and resistance affect campaign effectiveness. Advertising sensitivity reduces the potential to leverage the free social network as an information dissemination tool, while resistance diminishes the impact of advertisements. When sensitivity rises or when consumers are less receptive to advertising, the manufacturer should increase the participation rate to sustain adequate advertising efforts. However, when sensitivity or resistance becomes excessively high and advertising turns unprofitable, both parties should

consider withdrawing from parts of the market, or, as a last resort, exiting entirely.

- The distribution of gross profit margins influences the affordability of advertising costs. When the retailer has sufficient incentive to advertise independently, cooperative advertising may be unnecessary. Otherwise, each party should bear a larger share of advertising costs as its profit margin increases.

This study has several limitations that warrant further investigation. First, primarily due to data access limitations arising from privacy and business confidentiality concerns, our study does not include empirical data from real-world social networks and case studies. The model could be significantly strengthened if practical data were available for validation and calibration. Second, a systematic analysis of how the graph topology of the social network influences the performance of the algorithm proposed in this paper is an intriguing avenue for future research. Third, most of our results are derived under the assumption that the retailer's gross profit margin is not excessively low compared to that of the manufacturer. If this condition does not hold, it raises the question of whether an efficient algorithm exists to compute the Stackelberg equilibrium. Fourth, we focus solely on the voter model and the cost-per-click (CPC) mechanism. Some other models of consumer communication and/or payment mechanisms, including cost-per-sale (CPS) method, merit consideration. Additionally, consumer loyalty, which can significantly impact advertising effectiveness, deserves more attention. The role of initial attitude distribution should also be considered, as it can influence consumers' responses to cooperative advertising efforts. Finally, the dynamics of advertising competition among multiple manufacturers and retailers provides a promising topic for future research.

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Received November 2024; revised September 2025; early access October 2025.