Inventory Management for Dual Sales Channels with Inventory-Level-Dependent Demand

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Abstract

We study the optimal inventory policy of vendor operating dual channels. Demand of each channel depends on inventory levels of both channels. We propose a multi-period stochastic dynamic programming model which shows that under mild conditions, the myopic inventory policy is optimal ~~in~~ the infinite horizon problem. We consider the case where vendor neglects effects of inventory levels and give conditions under which total inventory of the system in this case is lower than optimal. Through numerical examples, we find that difference between inventory levels under these two cases can be very large, so effects of inventory levels are not negligible. Meanwhile, we do numerical examples to compare optimal inventory levels under centralized and decentralized ~~control~~, observing that ~~vendors~~ not always order ~~higher~~ under decentralized control. Besides, we do sensitivity analysis of the optimal inventory levels and service levels and examine effect of different ways to treat unmet demand ~~numerically~~.

Introduction

Nowadays, customers can buy products through many channels, e.g., physical stores, online stores, through telephone or by email. Among these channels, physical store and online store are often used. Different channels have their own advantages and disadvantages. In physical stores, customers can feel and touch products physically, but need a travel cost to go to the stores. On the ~~internet~~, customers can easily access stores online with convenience to obtain information about products, but they can't acquire products immediately. Therefore, different channels can satisfy different preferences of customers. Some customers prefer a particular kind of stores, physical stores or online stores. Other than fixed on a particular kind of stores, there are also customers dynamically choosing channels according to different prices and service levels (e.g., fill rate, sales effort).

To satisfy different needs of customers, many vendors establish online stores as a supplement to physical stores, like Gome and Suning, which are two largest household appliances sales companies in China. And many vendors open up physical stores on the basis of online stores, such as the ~~internet~~ pure-player iParty.com in the USA who has opened 52 physical stores. Multi-channel retailing can offer a better service to customers and bring with a higher customer loyalty (Wallace et al. (2004)). Meanwhile, it faces some challenges in operations along with opportunities, as stated by Müller-Lankenau et al. (2004) :"a company's activities in one channel influence a customer's decision on whether and how to use another channel". So the vendor should take the interaction of dual channels into account when integrating physical and online channels.

In this paper, we ~~want to~~ study the inventory management problem of vendor operating dual sales channels. ~~Researchers~~ assume demands in every period are independent identically distributed and are not influenced by inventory level ~~in traditional literature on inventory management~~. However, many evidences show that inventory level can affect demand. On one hand, more products on the shelf may attract the attention of customers more easily. In the review by Urban (2005), it is stated:" we often see mass displays of items in stores that are used as 'physical stock' (Larson and DeMarais (1990)) to stimulate sales of some retail items...... thus, increased inventory levels give the customer a wider selection and increase the probability of making a sale". On the other hand, we think that, a higher inventory level provides a higher service level (i.e., fill rate) with higher degree of customer satisfaction, thus leads to a larger demand.

Although there are models on "inventory-level-dependent demand rate" (i.e., demand rate depends on the inventory level and the higher inventory level is, the larger demand rate is), they only consider the inventory management of one channel. ~~We want to study~~ the inventory problem of operating dual channels. We propose a kind of demand which we call "inventory-level-dependent demand" and examine its effect on the structure of optimal inventory policy and optimal inventory levels ~~in this paper.~~

We formulate a multi-period stochastic dynamic programming model and show that under weak conditions there exists an optimal myopic policy for the infinite horizon problem. So the inventory problem of the vendor can be simplified to a single period problem and we obtain analytical ~~formulations~~ of optimal inventory levels ~~of~~ dual channels. Based on this result, we investigate the impact of inventory-level-dependent demand on the optimal inventory levels and service levels of dual channels. Meanwhile, we compare the optimal inventory levels to those of a system without consideration of effects of inventory levels and give conditions under which the optimal total inventory level in the former one is higher than that in the latter one. Through numerical example, we show that the difference of optimal inventory levels ~~in the~~ two cases ~~may~~ be very large, so effects of inventory levels are not negligible. After that, we run numerical examples to compare optimal inventory levels under centralized control and decentralized control. The results show that vendors tend to order more under decentralized control when inventory competition between dual channels is ~~not strong~~. For the cases with strong inventory competition, vendors may enlarge the inventory level of dual channels under centralized control to reduce competing effect. Furthermore, we examine how different ways of treating unmet demand affect optimal inventory levels. We consider three treatment mechanisms. ~~The first one is~~ unmet demand in physical store is lost and in online store it is backlogged. ~~The second one is~~ unmet demands are lost in both channels (lost sales case). ~~The last one is~~ unmet demands in both channels are both backlogged (backorder case). The optimal inventory levels under lostsales case are always higher than the levels under backorder case and the distance between optimal inventory levels in the first mechanism is always the largest.

Literature Review

There are three streams of literature related to our study. The first one is on inventory-level-dependent demand rate and inventory competition. The second one is on multi-location and multi-item inventory problems. The last one is on inventory distribution problem in a multi-channel system.

First, we briefly review the literature on inventory-level-dependent demand rate. As mentioned in Introduction, Urban (2005) gives a comprehensive review on the inventory models with inventory-level-dependent demand rate. The relevant papers regard time as continuous with demand rate being deterministic. The traditional approach to solve this kinds of models uses the time domain to generate a differential equation which represents the inventory level during the cycle (e.g., Baker and Urban (1988)). However, solving the differential equation is possible only for very special demand rate functions (e.g., the power function in Urban (2005)). Our assumption is different from theirs. In our model, time is discrete and demand is stochastic with relationship to inventory levels of dual channels after replenished. Some researchers have provided empirical evidences on inventory-level-dependent demand rate. Wolfe (1968) presents empirical evidence of the relationship between sales and inventory, noting that the sales of style merchandize, such as women's dresses of sports clothes, are proportional to the amount of inventory displayed. Koschat (2008) represents empirical evidence that demand indeed varies with inventory in the magazine retailing. Papers in this area assume or indicate that demand rate is proportional to the inventory displayed. However, we think even if not all the inventory we ordered is displayed, demand may increase with the inventory level, because higher inventory level brings higher fill rate and higher service level, thus higher demand. On the other hand, many papers study inventory competition among multiple newsvendors. Lippman and McCardle (1997) discuss a competitive newsvendor problem, in which the total demand is allocated among multiple newsvendors under certain splitting rules. They find that due to the demand-stealing effect, i.e., the more the newsvendor orders, the less the other newsvendors' demand stochastically, competition among multiple newsvendors results in supply chain inventory overstocking. Cachon (2003)(section 5) studies a similar newsvendor game but with a proportional demand allocation rule. He also finds that demand stealing effect leads to inventory overstocking. Charles (2010) extends the risk-neutral assumption in Cachon (2003) to risk-averse and shows that while the demand-stealing effect increases the total order quantity of the newsvendors, loss aversion effect decreases the newsvendors' total order quantity and if strong enough, may lead to a lower total inventory level of the decentralized supply chain than that of an integrated supply chain. In our model, dual channels compete through inventory levels. So, to integrate dual channels, the vendor should take the competing effect of inventory levels into account when making replenishment decisions.

The second stream of literature related to our paper is on multi-item and multi-location inventory problems. Veinott (1965) discusses a multi-product dynamic nonstationary inventory problem with periodic review. The author gives conditions to ensure the optimality of base stock policy. Ignall and Veinott (1969) analyze the same problem with relaxed conditions. Johnson (1967) considers a multi-item inventory problem system with periodic review and set-up cost. Demand in every period is assumed to depend on the stock level at the beginning of the period. The infinite horizon optimal policy is (σ, S) policy, which orders up to S for any point x (the stock level at the beginning of the period) in the reorder region σ and do not order for x not in σ. Because of the complexity of the problem, we don't take set-up cost into account. Our research is closely related to Ignall and Veinott (1969). We use the method in it to prove the optimality of myopic inventory policy. The difference between our research and theirs is that we assume demands of dual channels are inventory-level-dependent and we explore the impact of inventory-level-dependent demand on optimal inventory levels and optimal service levels. Kalin (1980) considers a standard period review, stochastic, dynamic multi-product inventory model with setup cost. He extends the results of Johnson (1967) and Wheeler (1968) by providing general conditions for the existence of an optimal (σ, S) policy. Ribinson (1990) examines a multi-period, multi-location inventory problem when allowing transhipment between retail outlet. Erkip et al. (1990) treat a depot-warehouse system, in which demand is allowed to be correlated with warehouses and also correlated in time. Eppen and Schrage (1981) model a depot-warehouse system with independent normally distributed stationary demand at the warehouses with lead times from supplier to the depot and from the depot to warehouses. The paper derives optimal parameters for given structure of the optimal policy. All these papers do not consider the effects of inventory levels on demand.

The third stream of literature related to our study is on inventory distribution problem in multi-channel system. The configuration of first part consists of a single vendor. Liang et al. (2011) compare site-to-store and store-to-site strategies for dual-channel integration. The site-to-store (or store-to-site) strategy can fill unmet orders in the physical channel (or online channel) from the inventory in the online channel (or physical channel). Alptekinoğlu and Tang (2005) develop a model of general multi-channel distribution system subject to stochastic demand. They consider a system distributing products to n sales locations through m cross-docking depots. Some papers (e.g. Agatz et al. (2008), Cattani et al. (2004)) analyze inventory rationing policy in multi-channel systems. They view different channels as different demand classes. While these policies are practical, they neglect the fact that "a company's activities in one channel influence a customer's decision on whether and how to use another channel" (Müller-Lankenau et al. (2004)). If one channel has a higher service level compared to other channels, customers buying from low-service-level channels may switch to the high-service-level channel. In this paper, we consider the effect of inventory level across channels. ~~The second part literature~~ is on inventory distribution problem in a multi-channel distribution system with a manufacturer and an independent retailer. Geng and Mallik (2007) consider the problem that a manufacturer distributes his product to the end consumer both through the independent retailer and his direct channel. If one channel is out of stock, a fraction of the unsatisfied customers visit the other channel, which induces inventory competition between the channels. Chen et al. (2008) study a manufacturer's problem of managing his direct online sales channel together with an independently owned retail channel. The two channels compete in service. Wu and Chiang (2011) discuss a system in which a wholesaler sells fashion products through two channels with asymmetric sales horizons. The sales horizon of e-tail channel is longer than that of a retailer channel. They suggest mechanisms to coordinate decisions of retailer and e-tailer in disjointed system. Seigfert et al. (2006) assume that the manufacturer has a direct market that serves a different customer segment with the traditional retail channel (not owned by the manufacturer) and that demands in the two channels are thus independent. They compare a dedicated supply chain to a cooperative supply chain.

Model and Structure of The Optimal Policy

The inventory system considered in this paper relates to a vendor selling goods to customers through dual channels: a physical store and an online store. The system is operated for multiple periods. Customer demands to both stores are stochastic. Revenues are received for satisfying customer demands. Unmet demands in the physical store are lost, whereas those in the online store are backlogged. Here, customers backlogged will pay at the time they are satisfied. Selling prices in dual channels are exogenously determined. The vendor makes replenishment decision at the beginning of each period, with zero lead time for replenishment. Also, we assume that no set-up cost is incurred for replenishment. The replenishment decision of the vendor is to determine inventory levels of dual channels. There is no inventory transhipment between dual channels. The objective of the vendor is to maximize the total profit of the entire system.

We introduce the following notations:
r_1: selling price in the physical store
r_2: selling price in the online store
h_1: unit inventory-holding cost in the physical store
h_2: unit inventory-holding cost in the online store
l_1: unit lostsales-penalty cost in the physical store
l_2: unit backorder-penalty cost in the online store
c_1: unit variable order cost for the physical store
c_2: unit variable order cost for the online store
γ: discount factor
M_1: capacity of warehouse for the physical store
M_2: capacity of warehouse for the online store
T: the number of periods during the decision horizon

Let, in a period, y_1 and y_2 be inventory levels of the two channels after replenishment, respectively. Referring to the idea in the literature (Urban (2005), Cachon (2003)), we assume that the demand in one channel increases with its inventory level. At the same time, we also consider that the demand in one channel may decrease with the inventory level of the other channel. That is, the two channels may compete for parts of customers. This should be reasonable for that some of customers may choose channels according to different prices and service levels (e.g., fill rate, sales effort), hence the two channels will compete for these customers. We adopt the following demand structure:

$$D_1(y_1, y_2) = a_1 y_1 - b_1 y_2 + \varepsilon_1, \tag{1}$$

$$D_2(y_1, y_2) = -a_2 y_1 + b_2 y_2 + \varepsilon_2. \tag{2}$$

In the above formulas, $\epsilon_1(\in [v_1, u_1])$ and $\epsilon_2(\in [v_2, u_2])$ are independent random variables with distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ respectively, representing demands of loyal customers to the individual channels. The demands of loyal customers are not influenced by inventory levels in either channels. Other than loyal customers, more demand to a particular channel is attracted with a linear relationship to its own inventory level (i.e. $a_1 y_1$ and $b_2 y_2$), and also a part of the demand is distracted to the other channel with a linear relationship to the other inventory level (i.e. $-b_1 y_2$ and $-a_2 y_1$). Hence, a_1 and b_2 indicate demand attraction ability of inventory levels, b_1 and a_2 indicate competing effect of inventory levels. To avoid trivial cases, we suppose that a_1, a_2, b_1 and b_2 all belong to (0, 1) and $b_1 \leq a_1$, $a_2 \leq b_2$, $a_1 + b_1 \leq 1$, $a_2 + b_2 \leq 1$. Moreover, in order to guarantee non-negative $D_1(y_1, y_2)$ and $D_2(y_1, y_2)$, we assume $b_1 \leq \frac{v_1}{M_2}$ and $a_2 \leq \frac{v_2}{M_1}$.

Denote, by $(x_1(t), x_2(t))$, the state of period t, which corresponds to initial inventories of dual channels in period t. Here, $x_1(t)$ is non-negative, but $x_2(t)$ can be either negative or nonnegative. Because no set-up cost is incurred for replenishment, the vendor can replenish inventory in every period without more cost caused from frequent set-up. Recall that unmet demand is backlogged at the online store. Therefore, the unmet demand must be satisfied in the next period to avoid more penalty cost. In doing so, the optimal inventory policy for the online store should be to replenish the inventory up to a nonnegative level.

Finite Horizon Problem

Suppose the system is operated for T periods. In period t ($1 \leq t \leq T$), on observed state (x_1, x_2), if the vendor replenishes inventory levels to y_1 and y_2, the revenues of dual channels are

$$r_1 E\min[y_1, D_1(y_1,y_2)], \tag{3}$$

and

$$r_2 E(\min[y_2, D_2(y_1,y_2)] + \gamma[x_2(t+1)]^-) = r_2 E[D_2(y_1,y_2)] - (1-\gamma)r_2 E[D_2(y_1,y_2)-y_2]^+, \tag{4}$$

respectively, where z^+ is the positive part of z, and z^- is the negative part of z. Revenue of the physical store (3) is obvious. Revenue of the online store, as shown at the left hand side of (4), includes two parts: the revenue from immediately satisfied demand in current period, and the revenue from unmet demand that is satisfied in next period. (Note that we discount revenue from the unmet demand received in next period to current period.)

Then, the resultant profits of dual channels are respectively

$$r_1 E\min[y_1, D_1(y_1,y_2)] - h_1 E[y_1 - D_1(y_1,y_2)]^+ - l_1 E[D_1(y_1,y_2)-y_1]^+ - c_1(y_1 - x_1), \tag{5}$$

and

$$\begin{aligned} &r_2 E[D_2(y_1,y_2)] - (1-\gamma)r_2 E[D_2(y_1,y_2)-y_2]^+ - h_2 E[y_2 - D_2(y_1,y_2)]^+ - l_2 E[D_1(y_1,y_2)-y_1]^+ \\ &- c_2(y_2 - x_2) \end{aligned} \tag{6}$$

In the next period, the state transfers to $([y_1 - D_1(y_1,y_2)]^+, y_2 - D_2(y_1,y_2))$.

Define

$$\begin{aligned} C(y_1,y_2) = \ & r_1 E\min[y_1, D_1(y_1,y_2)] - h_1 E[y_1 - D_1(y_1,y_2)]^+ - l_1 E[D_1(y_1,y_2)-y_1]^+ \\ & + r_2 E[D_2(y_1,y_2)] - ((1-\gamma)r_2 + l_2)E[D_2(y_1,y_2)-y_2]^+ - h_2 E[y_2 - D_2(y_1,y_2)]^+ \\ & - c_1 y_1 - c_2 y_2. \end{aligned}$$

Let, for given state (x_1,x_2) at period t, $V_t(x_1,x_2)$ be the expected profit-onward from period t by the optimal inventory policy.

We assume

$$V_{T+1}(x_1,x_2) = c_1 x_1 + c_2 x_2 \tag{7}$$

for the end period.

The optimality equations can be expressed as follows:

$$V_t(x_1,x_2) = c_1 x_1 + c_2 x_2 + \max_{x_1 \le y_1 \le M_1, x_2^+ \le y_2 \le M_2} \{H_t(y_1,y_2)\}, \ 1 \le t \le T, \tag{8}$$

$$H_t(y_1,y_2) = \ C(y_1,y_2) + \gamma E V_{t+1}\left([y_1 - D_1(y_1,y_2)]^+, y_2 - D_2(y_1,y_2)\right), \ 1 \le t \le T. \tag{9}$$

The following lemma is straightforward from Topkis (1968).

Lemma 1 *If $f(u,\omega)$ is a convex real-valued function on the convex set A, then $g(u) = inf_{\omega:(u,\omega)\in A} f(u,\omega)$ is convex on $\{u : \text{there exists } (u,\omega) \in A\}$.*

Then, we have the following result.

Theorem 1 *(1) $H_t(y_1,y_2)$ is concave in y_1 and y_2 for all $1 \le t \le T$;*
(2) $V_t(x_1,x_2)$ is concave and decreasing in x_1 and x_2 for all $1 \le t \le T+1$.

See proof in the appendix. Because replenishment decisions of dual channels are correlated, there is no simple structure of the optimal inventory policy in finite horizon problem. Nevertheless, Theorem 1 can be useful in calculating the optimal policy.

Infinite Horizon Problem

For the model of finite horizon problem, we can rewrite optimality equations (7) to (9). By referring to the method in chapter 9.4.2 of Zipkin (2000), we define

$$V_t^+(x_1, x_2) = -c_1x_1 - c_2x_2 + V_t(x_1, x_2), \qquad 1 \le t \le T + 1,$$

and

$$\begin{aligned} C^+(y_1, y_2) = &- (r_1 + l_1)E[D_1(y_1, y_2) - y_1]^+ - (h_1 - \gamma c_1)E[y_1 - D_1(y_1, y_2)]^+ + r_1E[D_1(y_1, y_2)] \\ &- ((1-\gamma)r_2 + l_2)E[D_2(y_1, y_2) - y_2]^+ - h_2E[y_2 - D_2(y_1, y_2)]^+ \\ &+ (r_2 - \gamma c_2)E[D_2(y_1, y_2)] - c_1y_1 - c_2(1-\gamma)y_2. \end{aligned}$$

Then, optimality equations (7) to (9) are transformed to the following recursion:

$$V_{T+1}^+(x_1, x_2) = 0, \tag{10}$$

$$H_t(y_1, y_2) = C^+(y_1, y_2) + \gamma E\{V_{t+1}^+([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))\}, \ 1 \le t \le T, \tag{11}$$

$$V_t^+(x_1, x_2) = max\{H_t(y_1, y_2) : x_1 \le y_1 \le M_1, x_2^+ \le y_2 \le M_2\}, \ 1 \le t \le T. \tag{12}$$

From (11), we can see that $C^+(y_1, y_2)$ is the system profit in a single period.

Lemma 2 *$C^+(y_1, y_2)$ is concave and $\frac{\partial^2}{\partial y_1 \partial y_2} C^+(y_1, y_2) = \frac{\partial^2}{\partial y_2 \partial y_1} C^+(y_1, y_2) \ge 0$ for any y_1 and y_2.*

For infinite horizon system, we consider a stationary inventory policy, denoted by (π_1, π_2). Here, π_1 specifies the replenishment rule of the physical store, by which the order-up-to level is determined for a given state, and π_2 has a similar meaning. In this subsection, we continue to use $C(y_1, y_2)$ and $C^+(y_1, y_2)$ defined previously.

Let $V(x_1, x_2|\pi_1, \pi_2)$ be the total expected discounted profit of dual channels over the infinite horizon for a given initial state $(x_1(1) = x_1,\ x_2(1) = x_2)$ when we use policy (π_1, π_2). We have

$$\begin{aligned} V(x_1, x_2|\pi_1, \pi_2) = \ & E\{\sum_{t=1}^{\infty} \gamma^t [c_1x_1(t) + c_2x_2(t) + C(y_1(t), y_2(t))]| \\ & x_1(1) = x_1, x_2(1) = x_2, x_1(t) \le y_1(t) \le M_1, (x_2(t))^+ \le y_2(t) \le M_2\}. \end{aligned} \tag{13}$$

Substituting $x_1(t+1) = [y_1 - D_1(y_1, y_2)]^+$ and $x_2(t+1) = y_2 - D_2(y_1, y_2)$ into (13) leads to

$$V(x_1, x_2|\pi_1, \pi_2) = c_1x_1 + c_2x_2 + E[\sum_{t=1}^{\infty} \gamma^t C^+(y_1(t), y_2(t))|x_1(t) \le y_1(t) \le M_1, (x_2(t))^+ \le y_2(t) \le M_2].$$

Recall that $C^+(y_1, y_2)$ is the profit of the system in a single period. Let $y^* = (y_1^*, y_2^*)$ be one of the maximizer of $C^+(y_1, y_2)$ on domain $\Omega = \{0 \le y_1 \le M_1, 0 \le y_2 \le M_2\}$.

Lemma 3 *If $1 - a_1 - b_2 + a_1b_2 - a_2b_1 \ne 0$ and $F_1(\cdot)$, $F_2(\cdot)$ both are strictly increasing distribution functions, y^* is unique.*

In the following, we assume $1 - a_1 - b_2 + a_1b_2 - a_2b_1 \ne 0$ and $F_1(\cdot)$, $F_2(\cdot)$ are both strictly increasing distribution functions (under these conditions, the optimal inventory policy will have a simple structure), in which condition $1 - a_1 - b_2 + a_1b_2 - a_2b_1 \ne 0$ holds with probability one.

Define functions

$$
\begin{aligned}
z_1(x_2) &= \arg \max_{0 \le y_1 \le M_1} C^+(y_1, x_2), \quad y_2^* < x_2 \le M_2, \\
z_2(x_1) &= \arg \max_{0 \le y_2 \le M_2} C^+(y_1, x_2), \quad y_1^* < x_1 \le M_1.
\end{aligned} \tag{14}
$$

Denote regions

$$
\begin{aligned}
&\Omega_1 = \{x \in R^2 : x \le y^*\}, \quad \Omega_2 = \{x \in R^2 : y_2^* < x_2 \le M_2, 0 \le x_1 < z_1(x_2)\}, \\
&\Omega_3 = \{x \in R^2 : y_1^* < x_1 \le M_1, x_2 < z_2(x_1)\}, \quad \textit{and } \Omega_4 = \{x \in R^2 : y_1^* \le x_1 \le M_1, y_2^* \le x_2 \le M_2\}.
\end{aligned} \tag{15}
$$

Lemma 4 *$z_1(x_2)$ and $z_2(x_1)$ are non-increasing functions.*

We define the following inventory policy:

$$
\bar{y}(x_1, x_2) = \begin{cases} y^*, & x \le y^*, \\ (x_1, z_2(x_1)), & x_1 > y_1^* \text{ and } x_2 < y_2^*, \\ (z_1(x_2), x_2), & x_1 < y_1^* \text{ and } x_2 > y_2^*, \\ (x_1, x_2), & x \ge y^*. \end{cases} \tag{16}
$$

The following theorem shows that $\bar{y}(x_1, x_2)$ is the myopic policy (maximizing the profit in current period) and gives conditions under which the myopic inventory policy is optimal for infinite horizon problem.

Theorem 2 *If $1 - a_1 - b_2 + a_1 b_2 - a_2 b_1 \neq 0$ and $F_1(\cdot)$, $F_2(\cdot)$ are both strictly increasing distribution functions, $\bar{y}(x_1, x_2)$ is the myopic inventory policy and it is optimal in the infinite horizon problem.*

The proof is given in appendix.

Properties

To obtain analytical results of the optimal inventory levels, we assume the initial inventories of dual channels in the first period are both zero from now on. The following corollary is straightforward from Theorem 2.

Corollary 1 *When initial inventories of dual channels in the first period are both zero, the optimal inventory policy is to order inventory up to y^* in every period.*

Now, the infinite horizon problem can be transformed to a single period problem as follows:

$$
\begin{aligned}
\text{maximize } & C^+(y_1, y_2) \\
s.t. \ & 0 \le y_1 \le M_1 \\
& 0 \le y_2 \le M_2.
\end{aligned} \tag{17}
$$

It is valuable to study the impact of inventory-level-dependent demand on the optimal inventory levels and service levels of dual channels. Suppose $(y_1^{'}, y_2^{'})$ is the maximizer of

$C^{+}(y_1, y_2)$. By taking derivatives of $C^{+}(y_1, y_2)$ with respect to y_1 and y_2, with incorporation of functions $F_1(\cdot)$ and $F_2(\cdot)$, we have

$$\begin{aligned}\frac{\partial}{\partial y_1}C^{+}(y_1,y_2) = & -(1-a_1)(r_1+l_1+h_1-\gamma c_1)F_1((1-a_1)y_1^{'}+b_1y_2^{'})-a_2((1-\gamma)r_2+l_2+h_2)\\ & F_2(a_2y_1^{'}+(1-b_2)y_2^{'})+r_1-c_1-\gamma a_2(r_2-c_2)+a_2l_2-a_1l_1+l_1=0,\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial}{\partial y_2}C^{+}(y_1,y_2) = & -b_1(r_1+l_1+h_1-\gamma c_1)F_1((1-a_1)y_1^{'}+b_1y_2^{'})-(1-b_2)((1-\gamma)r_2+l_2+h_2)\\ & F_2(a_2y_1^{'}+(1-b_2)y_2^{'})+(1-\gamma+\gamma b_2)(r_2-c_2)+l_1b_1-l_2b_2+l_2=0.\end{aligned} \tag{19}$$

Solving $F_1((1-a_1)y_1^{'}+b_1y_2^{'})$ and $F_2(a_2y_1^{'}+(1-b_2)y_2^{'})$ from (18) and (19) obtains

$$F_1((1-a_1)y_1^{'}+b_1y_2^{'}) = \frac{l_1}{k_1}+\frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}, \tag{20}$$

$$F_2(a_2y_1^{'}+(1-b_2)y_2^{'}) = \frac{l_2-\gamma(r_2-c_2)}{k_2}+\frac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2}, \tag{21}$$

where $k_1 = r_1+l_1+h_1-\gamma c_1$ and $k_2 = (1-\gamma)r_2+l_2+h_2$.

Let

$$A = (r_1-c_1)(1-b_2)-(r_2-c_2)a_2,$$
$$B = (r_2-c_2)(1-a_1)-(r_1-c_1)b_1.$$

In practice, $1-b_2$ should be much larger than a_2 and $1-a_1$ should be much larger than b_1. So, provided that the profit margin (r_1-c_1) of the physical store is not significantly different from that of the online store (r_2-c_2), $A \geq 0$ and $B \geq 0$ always hold. In the following, we assume $A \geq 0$ and $B \geq 0$. Then, the right hand side of equations (20) and (21) are larger than 0.

On the other hand, if the right hand side of equations (20) or (21) are larger than 1, $y_1^{'}$ or $y_2^{'}$ are infinite. This case is very trivial. So, we only consider cases in which the right hand sides of equations (20) and (21) are between 0 and 1.

From (20) and (21), we have

$$y_1^{'} = \frac{(1-b_2)F_1^{-1}(\frac{l_1}{k_1}+\frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1})-b_1F_2^{-1}(\frac{l_2-\gamma(r_2-c_2)}{k_2}+\frac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2})}{(1-a_1)(1-b_2)-a_2b_1}, \tag{22}$$

$$y_2^{'} = \frac{(1-a_1)F_2^{-1}(\frac{l_2-\gamma(r_2-c_2)}{k_2}+\frac{(r_2-c_2)(1-a_1)-(r_1-c_1)b_1}{[(1-a_1)(1-b_2)-a_2b_1]k_2})-a_2F_1^{-1}(\frac{l_1}{k_1}+\frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1})}{(1-a_1)(1-b_2)-a_2b_1}. \tag{23}$$

To avoid trivial cases, we assume that $y_1^{'}$ and $y_2^{'}$ are nonnegative. In practice, $1-b_2$ is much larger than b_1 and $1-a_1$ is much larger than a_2. So, provided that the cost factors (r_1, r_2 and l_1, l_2 etc) and scale of loyal customers (u_1, u_2 and v_1, v_2) of dual channels are not significantly different from each other, $y_1^{'}$ and $y_2^{'}$ are nonnegative. In the following, we study cases where $(y_1^{'}, y_2^{'})$ satisfies constraints $0 \leq y_1^{'} \leq M_1$ and $0 \leq y_2^{'} \leq M_2$. Then, $y_1^{*} = y_1^{'}$, $y_2^{*} = y_2^{'}$.

Sensitivity Analysis

The following proposition states how the optimal inventory levels change as demand attraction ability of inventory level increases.

Proposition 1 *(1) y_1^{*} is increasing with a_1, y_2^{*} is decreasing with a_1;*
(2) y_1^{*} is decreasing with b_2, y_2^{*} is increasing with b_2.

Now, we calculate the optimal service levels of dual channels. Here, the service level denotes the probability of an arbitrarily arriving customer being served from stock on hand, i.e., Prob{number of demands in a period $\leq$ inventory-on-hand at the beginning of the period}.

The optimal service level of the physical store is

$$s_1^* = P\{y_1^* \geq D_1(y_1^*, y_2^*)\} = P\{(1-a_1)y_1^* + b_1 y_2^* \geq \varepsilon_1\}. \tag{24}$$

Substitute equations (22) and (23) into the above, and note that $F_1(\cdot)$ is the cumulative distribution of ε_1, we have

$$s_1^* = \frac{l_1}{k_1} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}. \tag{25}$$

Similarly, the optimal service level of the online store is given by

$$s_2^* = P\{y_2^* \geq D_2(y_1^*, y_2^*)\} = \frac{l_2-\gamma(r_2-c_2)}{k_2} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_1)}{[(1-a_1)(1-b_2)-a_2b_1]k_2}. \tag{26}$$

From equations (25) and (26), we can easily obtain the following proposition.

Proposition 2 *(1) s_1^* (s_2^*) increases with the profit margin of the physical store (online store) and decreases with the profit margin of the online store (physical store);*
(2) s_1^* increases with a_1 and b_1, s_2^* decreases with a_1 and b_1;
(3) s_1^* decreases with a_2 and b_2, s_2^* increases with a_2 and b_2.

It is known, from Proposition 1, that y_1^* increases with a_1. Note that demand of physical channel that is D_1 also increases with a_1. So "s_1^* increases with a_1" in (2) in Proposition 2 implies that increment of y_1^* is faster than D_1 as a_1 increases. When b_1 increases, y_1^* may increase or decrease and D_1 decreases. When y_1^* increases, s_1^* must increase as b_1 increases. When y_1^* decreases with b_1, "s_1^* decreases with b_1" in (2) in Proposition 2 implies that decrement of y_1^* must be slower than decrement of D_1 as b_1 increases. Interpretations of other parts in Proposition 2 are similar.

What if the vendor neglects effects of inventory levels?

In this subsection, we examine the effect of inventory-level-dependent demand on the optimal inventory levels. What if the vendor neglects effects of inventory levels? In this case, he regards parameters a_1, b_1, a_2 and b_2 in demand formulas (1) and (2) as zero. That is, demands in two channels are ϵ_1 and ϵ_2 respectively. The inventory problem of the system are simplified to two independent newsvendor problems. So the optimal inventory levels of dual channels are $y_1^{''} = F_1^{-1}(\frac{r_1-c_1+l_1}{k_1})$ and $y_2^{''} = F_2^{-1}(\frac{(1-\gamma)(r_2-c_2)+l_2}{k_2})$, with corresponding service levels being $s_1^{''}$ and $s_2^{''}$.

The following proposition compares optimal inventory levels in this case with the optimal decisions when taking effects of inventory levels into account.

Let

$$\begin{aligned}
C &= (r_1-c_1)[(1-b_2)a_1 + a_2b_1] - (r_2-c_2)a_2, \\
D &= (r_2-c_2)[(1-a_1)b_2 + a_2b_1] - (r_1-c_1)b_1, \\
e_1 &= a_1 - a_2 - (a_1b_2 - a_2b_1), \\
e_2 &= b_2 - b_1 - (a_1b_2 - a_2b_1).
\end{aligned}$$

Proposition 3 *(1) When $C \geq 0$ and $D \geq 0$, it follows that $y_2^* \geq y_2^{''}$ if $y_1^* \leq y_1^{''}$; and $y_1^* \geq y_1^{''}$ if $y_2^* \leq y_2^{''}$.*
(2) When $C \geq 0$, $D \geq 0$, $e_1 \geq 0$ and $e_2 \geq 0$, it follows that $y_1^* + y_2^* \geq y_1^{''} + y_2^{''}$.

When $r_1 - c_1 = r_2 - c_2$, condition $C \geq 0$ is equivalent to $e_1 \geq 0$ and $D \geq 0$ is equivalent to $e_2 \geq 0$. Conditions $e_1 \geq 0$ and $e_2 \geq 0$ indicate the demand attraction ability of inventory levels in the system is heavier than the competing effect of inventory levels in the system. Furthermore, when $e_1 \geq 0$ and $e_2 \geq 0$, conditions $C \geq 0$ and $D \geq 0$ indicate increasing total inventory level is beneficial. So, the vendor should order more total inventory for dual channels if he takes effects of inventory level into account. Sometimes, the optimal inventory level of one channel may be lower than the level with which the vendor neglects effects of inventory levels; this must appear under the consideration of reducing the competing effect with the demand of the other channel. So, the optimal inventory level of the other channel must be higher than the level with which the vendor neglects effects of inventory levels.

b_1	y_1^*	y_2^*	e_1	e_2	C	D	s_1^*	s_2^*
0.02	120.0838	45.0226	0.1310	0.0610	0.8360	0.2850	0.9393	0.8842
0.03	119.5479	44.7116	0.1315	0.0515	0.8390	0.2275	0.9396	0.8739
0.04	119.0199	44.3996	0.1320	0.0420	0.8420	0.1700	0.9398	0.8637
0.05	118.4998	44.0867	0.1325	0.0325	0.8450	0.1125	0.9401	0.8534
0.06	117.9876	43.7729	0.1330	0.0230	0.8480	0.0550	0.9403	0.8432
0.07	117.4833	43.4582	0.1335	0.0135	0.8510	-0.0025	0.9406	0.8329
0.08	116.9870	43.1426	0.1340	0.0040	0.8540	-0.0600	0.9408	0.8226
0.09	116.4987	42.8260	0.1345	-0.0055	0.8570	-0.1175	0.9411	0.8123
0.1	116.0185	42.5086	0.1350	-0.0150	0.8600	-0.1750	0.9413	0.8020
0.11	115.5464	42.1901	0.1355	-0.0245	0.8630	-0.2325	0.9416	0.7916
0.12	115.0824	41.8708	0.1360	-0.0340	0.8660	-0.2900	0.9418	0.7813
0.13	114.6265	41.5505	0.1365	-0.0435	0.8690	-0.3475	0.9421	0.7709
0.14	114.1789	41.2293	0.1370	-0.0530	0.8720	-0.4050	0.9423	0.7605
0.15	113.7394	40.9071	0.1375	-0.0625	0.8750	-0.4625	0.9426	0.7501
0.16	113.3083	40.5840	0.1380	-0.0720	0.8780	-0.5200	0.9428	0.7397
0.17	112.8854	40.2599	0.1385	-0.0815	0.8810	-0.5775	0.9431	0.7293
0.18	112.4709	39.9349	0.1390	-0.0910	0.8840	-0.6350	0.9433	0.7188
0.19	112.0647	39.6089	0.1395	-0.1005	0.8870	-0.6925	0.9435	0.7084
0.20	111.6670	39.2820	0.1400	-0.1100	0.8890	-0.7500	0.9438	0.6979

Table 1: Optimal inventory levels and service levels under different values of b_1

We give a numerical example to compare optimal inventory levels respectively with and without effects of inventory levels to demands. The parameters are as follows: $a_1 = 0.2$, $b_1 = 0.02 : 0.01 : 0.2$, $a_2 = 0.05$, $b_2 = 0.1$, $r_1 = 10$, $r_2 = 8$, $c_1 = 4$, $c_2 = 3$, $h_1 = 2$, $h_2 = 1$, $l_1 = 12$, $l_2 = 6$, $\gamma = 0.9$, $M_1 = 200$, $M_2 = 100$, ϵ_1 is uniformly distributed on [50, 100], ϵ_2 is uniformly distributed on [20, 50]. When the vendor neglects effects of inventory level, he orders inventory up to levels $y_1^{''} = 94.12$ and $y_2^{''} = 45$ with corresponding service levels being $s_1^{''} = 0.8824$ and $s_2^{''} = 0.8333$. The optimal inventory levels and service levels are represented in Table 1.

Furthermore, we have the following three observations:

(1) With $A = (r_1 - c_1)(1 - b_2) - (r_2 - c_2)a_2 = 4.4 \geq 0$ and $B = (r_2 - c_2)(1 - a_1) - (r_1 - c_1)b_1 \geq 0$, it is known that s_1^* increases with b_1 and s_2^* decreases with b_1. This is consistent with proposition 2.

(2) From Table 1, we can see that $y_1^* \geq y_1^{''}$ when $y_2^* \leq y_2^{''}$. Moreover, $y_1^* + y_2^* \geq y_1^{''} + y_2^{''}$ always holds. These also demonstrate that conditions in Proposition 3 are sufficient conditions.

(3) From Table 1, we observe that the differences between y_1^* and $y_1^{''}$, y_1^* and $y_1^{''}$ are very large. If the vendor neglects the effects of inventory levels, the vendor may order as much as 21.6 percents lower than optimal for the physical store or 14.6 percents higher than optimal for the online store. The vendor will lost a large proportion of profit if he neglects the effects of inventory levels. So the effects of inventory levels are not negligible.

Discussions

Comparison between Centralized Control and Decentralized Control

In the previous sections, the physical store and the online store are managed under retailer's centralized control. Another scenario is that dual channels are operated by separate organizations. As Zhang et al. (2010) describes: "the channels may have different target markets requiring unique merchandise and pricing. Due to these operational differences, many multichannel retailers have separate organizations for each channel and even outsource channel management, which further increases the challenges in achieving demand synergies". We call this case decentralized control. When dual channels are operated by different organizations, they will compete with each other through inventory. We denote the equilibrium of inventory as $y_1^{'''}$ and $y_2^{'''}$.

The profits of the two organizations in every period are given by:

$$-(r_1 + l_1)E[D_1(y_1, y_2) - y_1]^+ - (h_1 - \gamma c_1)E[y_1 - D_1(y_1, y_2)]^+ + r_1 E[D_1(y_1, y_2)] - c_1 y_1$$

and

$$-((1-\gamma)r_2 + l_2)E[D_2(y_1, y_2) - y_2]^+ - h_2 E[y_2 - D_2(y_1, y_2)]^+ + (r_2 - \gamma c_2)E[D_2(y_1, y_2)] - c_2(1-\gamma)y_2$$

respectively. Then, it is not difficult to obtain:

$$y_1^{'''} = \frac{(1-b_2)F_1^{-1}(\frac{l_1}{k_1} + \frac{r_1 - c_1}{(1-a_1)k_1}) - b_1 F_2^{-1}(\frac{l_2 - \gamma(r_2 - c_2)}{k_2} + \frac{r_2 - c_2}{(1-b_2)k_2})}{(1-a_1)(1-b_2) - a_2 b_1}, \tag{27}$$

$$y_2^{'''} = \frac{(1-a_1)F_2^{-1}(\frac{l_2 - \gamma(r_2 - c_2)}{k_2} + \frac{r_2 - c_2}{(1-b_2)k_2}) - a_2 F_1^{-1}(\frac{l_1}{k_1} + \frac{r_1 - c_1}{(1-a_1)k_1})}{(1-a_1)(1-b_2) - a_2 b_1}. \tag{28}$$

Using the same parameter values as in the numerical example of the Properties section, we compare y_1^*, y_2^* with $y_1^{'''}$, $y_2^{'''}$ under different values of b_1 and list them in Table 2. We observe that, vendors not always order higher under decentralized control; in most cases, vendors tend to order more under decentralized control than under centralized control. This phenomenon can be explained by the demand-stealing effect, that is, the higher the vendor orders, the less the other channel demand is, so vendors tend to order more inventory under decentralized control than under centralized control. When b_1 is large ($b_1 \geq 0.15$), which indicates competition is intense, we can see that $y_1^* \geq y_1^{'''}$, $y_2^* \leq y_2^{'''}$, so $y_1^* - y_2^* \geq y_1^{'''} - y_2^{'''}$. Hence, when competition between dual channels is intense, vendors will enlarge the distance between the inventory levels of dual channels to reduce the competing effect under centralized control. From table 2, we observe that vendors order as much as 17.7 percents higher for the online store under decentralized control than under centralized control. So vendors should think up mechanisms to coordinate dual channels when they belong to the same company.

The Effect of Different Ways of Treating Unmet Demand

In the previous sections, we assume that unmet demand in the physical store is lost and in the online store it is backlogged. Now, we investigate how different ways of treating unmet demand affect optimal inventory levels of dual channels. The system is operated under centralized control. We consider two other treatment mechanisms. One is that unmet demands in both channels are lost (we call this lostsales case) and the other one is that unmet demands are backlogged in both channels (we call this backorder case). Suppose that the optimal inventory levels corresponding to these two cases are (y_1^{**}, y_2^{**}) and (y_1^{***}, y_2^{***}) respectively. Using the same method as in the Properties section , we have

$$y_1^{**} = \frac{(1-b_2)F_1^{-1}(\frac{l_1}{k_1} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1}) - b_1F_2^{-1}(\frac{l_2}{k_2'} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_2)}{[(1-a_1)(1-b_2)-a_2b_1]k_2'})}{(1-a_1)(1-b_2)-a_2b_1}, \quad (29)$$

$$y_2^{**} = \frac{(1-a_1)F_2^{-1}(\frac{l_2}{k_2'} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_2)}{[(1-a_1)(1-b_2)-a_2b_1]k_2'}) - a_2F_1^{-1}(\frac{l_1}{k_1} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1})}{(1-a_1)(1-b_2)-a_2b_1}, \quad (30)$$

$$y_1^{***} = \frac{(1-b_2)F_1^{-1}(\frac{l_1-\gamma(r_1-c_1)}{k_1'} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1'}) - b_1F_2^{-1}(\frac{l_2-\gamma(r_2-c_2)}{k_2} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_2)}{[(1-a_1)(1-b_2)-a_2b_1]k_2})}{(1-a_1)(1-b_2)-a_2b_1}, \quad (31)$$

$$y_2^{***} = \frac{(1-a_1)F_2^{-1}(\frac{l_2-\gamma(r_2-c_2)}{k_2} - \frac{(r_1-c_1)b_1-(r_2-c_2)(1-a_2)}{[(1-a_1)(1-b_2)-a_2b_1]k_2}) - a_2F_1^{-1}(\frac{l_1-\gamma(r_1-c_1)}{k_1'} + \frac{(r_1-c_1)(1-b_2)-(r_2-c_2)a_2}{[(1-a_1)(1-b_2)-a_2b_1]k_1'})}{(1-a_1)(1-b_2)-a_2b_1}, \quad (32)$$

b_1	y_1^*	$y_1^{'''}$	y_2^*	$y_1^{'''}$
0.02	120.0838	121.1015	45.0226	45.6463
0.03	119.5479	120.5297	44.7116	45.6781
0.04	119.0199	119.9572	44.3996	45.7099
0.05	118.4998	119.3838	44.0867	45.7417
0.06	117.9876	118.8096	43.7729	45.7736
0.07	117.4833	118.2347	43.4582	45.8056
0.08	116.9870	117.6589	43.1426	45.8376
0.09	116.4987	117.0823	42.8260	45.8696
0.1	116.0185	116.5049	42.5086	45.9017
0.11	115.5464	115.9268	42.1901	45.9338
0.12	115.0824	115.3478	41.8708	45.9660
0.13	114.6265	114.7679	41.5505	45.9982
0.14	114.1789	114.1873	41.2293	46.0304
0.15	113.7394	113.6059	40.9071	46.0627
0.16	113.3083	113.0236	40.5840	46.0951
0.17	112.8854	112.4406	40.2599	46.1275
0.18	112.4709	111.8567	39.9349	46.1599
0.19	112.0647	111.2720	39.6089	46.1924
0.20	111.6670	110.6864	39.2820	46.2249

Table 2: Optimal inventory levels under centralized control and decentralized control

where $k_1^{'} = r_1 + l_1 + h_1 - \gamma c_1$ and $k_2^{'} = r_2 + l_2 + h_2 - \gamma c_2$. In (29) and (30), l_1 and l_2 denote unit lostsales-penalty cost in dual channels respectively, while in formulations (31) and (32) they represent unit penalty backorder-cost of dual channels respectively.

b_1	y_1^*	y_1^{**}	y_1^{***}	y_2^*	y_2^{**}	y_2^{***}
0.02	120.0838	120.0334	116.8400	45.0226	47.0392	45.2028
0.03	119.5479	119.4656	116.3149	44.7116	46.9076	44.8912
0.04	119.0199	118.9011	115.7978	44.3996	46.7756	44.5786
0.05	118.4998	118.3400	115.2885	44.0867	46.6433	44.2651
0.06	117.9876	117.7823	114.7872	43.7729	46.5105	43.9507
0.07	117.4833	117.2279	114.2939	43.4582	46.3773	43.6354
0.08	116.9870	116.6769	113.8086	43.1426	46.2437	43.3192
0.09	116.4987	116.1293	113.3314	42.8260	46.1096	43.0020
0.1	116.0185	115.5852	112.8622	42.5086	45.9751	42.6839
0.11	115.5464	115.0445	112.4012	42.1901	45.8403	42.3649
0.12	115.0824	114.5073	111.9483	41.8708	45.7050	42.0449
0.13	114.6265	113.9735	111.5036	41.5505	45.5692	41.7240
0.14	114.1789	113.4432	111.0671	41.2293	45.4331	41.4022
0.15	113.7394	112.9164	110.6389	40.9071	45.2965	41.0794
0.16	113.3083	112.3932	110.2190	40.5840	45.1595	40.7556
0.17	112.8854	111.8735	109.8075	40.2599	45.0220	40.4309
0.18	112.4709	111.3573	109.4043	39.9349	44.8842	40.1053
0.19	112.0647	110.8447	109.0095	39.6089	44.7458	39.7787
0.20	111.6670	110.3357	108.6231	39.2820	44.6071	39.4511

Table 3: Optimal inventory levels under three different mechanism of treating unmet demand

From many numerical examples, we observe the following properties:

$$\begin{aligned} y_1^* \geq & y_1^{**} \geq y_1^{***} \\ & y_2^{**} \geq y_2^{***} \geq y_2^* \end{aligned} \tag{33}$$

Table 3 lists the result of an example. In this example the unit lostsales-penalty cost and backorder-penalty cost in physical store are 12 and 8 respectively and in online store they are 10 and 6 respectively. Other parameters are the same as in the numerical example in the Properties section. Optimal inventory levels under lostsales case are always higher than those under backorder case. This can be explained as follows: because unmet demand under the backorder case can be satisfied in the next period, the vendor sets lower inventory levels for dual channels compared to the lostsales case to reduce the competing effect of inventory levels.

When unmet demand in physical store is lost and in online store it is backlogged, the vendor raises up inventory level of physical store and reduces inventory level of online store compared to the lostsales case and backorder case. This is because unmet demand in physical store will be lost and in online store it will be satisfied in the next period, so the vendor provides a higher service level in the physical store and decreases service level of online store properly to reduce its competing effect with physical store.

Conclusion

In this paper, we study the inventory policy of a vendor operating dual sales channels. Demands of dual channels are inventory-level-dependent, increasing with inventory level of its own channel and decreasing with inventory level of the other channel. We prove that the myopic inventory policy is optimal for the infinite horizon problem. When the initial inventories of dual channels in the first period are zero, we can simplify the infinite horizon problem into a single period problem and obtain formulations of the optimal inventory levels of dual channels. So it is easy for us to investigate the impact of inventory-level-dependent demand on the optimal inventory levels and the optimal service levels of dual channels. We also consider the case where the vendor neglects the effects of inventory levels and compare inventory levels in this case with the optimal levels. Through numerical examples, we show that the difference of the optimal inventory levels of dual channels in the two cases may be very large, so the vendor may lost a large proportion of profit if he neglects effects of inventory levels. For a more in-depth discussion, we compare the optimal inventory levels under decentralized control and centralized control. Through numerical examples, we find that when inventory competition is weak, vendors tend to order more under decentralized control and when competition is intense, vendors will enlarge the distance of inventory levels between dual channels to reduce competition under centralized control; the difference between the optimal levels under centralized control and decentralized control may be very large and the vendors should think up mechanisms to integrate dual channels. After that, we investigate how different ways of treating unmet demand affect optimal inventory levels, and find that optimal inventory levels under lostsales case are always higher than those under backorder case and the difference between inventory levels of dual channels is the largest when unmet demand in the physical store is lost and in online store is backlogged.

There are three directions for future research. First, demands of dual channels are inventory-level-dependent in our paper, linearly increasing with inventory level of its own channel and linearly decreasing with inventory level of the other channel. Nonlinear types of inventory-level-dependent demand models may be also considered in multi-channel inventory problem. Second, in this paper, the vendor's inventory decision only affects demand in current period. But the decision in one period may affect demands thereafter. Azadivar et al. (2010) consider the discrete multi-period dynamic inventory control problem where customers follow a simple satisfaction-based demand process; the probability of demand depends on whether their demand was satisfied last time. Popescu and Wu (2007) study dynamic pricing problem of a monopolist firm in a market with repeated interactions, where demand is sensitive to the firm's pricing history. So, as one of future directions, we can consider models in which demand depends on firm's inventory on history. More reasonably, demand depends on firm's service level on history. Third, analysis of a system with setup cost is also valuable.

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APPENDIX

Proof of Theorem 1.

We prove this theorem by induction on t. It holds obviously when $t = T + 1$. Suppose it holds for $t = k + 1$, i.e., $V_{k+1}(x_1, x_2)$ is concave and decreasing in x_1 and x_2. We need to show that $V_k(x_1, x_2)$ is concave and decreasing in x_1 and x_2.

It is easy to see that the decision space $\{x_1 \leq y_1 \leq M_1, x_2^+ \leq y_2 \leq M_2\}$ is a convex set. So, it suffices, by ~~lemma~~ 1, to show $H_k(y_1, y_2)$ is concave in y_1 and y_2. Substituting expressions of $D_1(y_1, y_2)$ and $D_2(y_1, y_2)$, i.e., equations (1) and (2), into $C(y_1, y_2)$, we can prove that every item in $C(y_1, y_2)$ is concave. Hence, $C(y_1, y_2)$ is concave.

Now, we prove the concavity of $EV_{k+1}([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))$. We just need to show that $V_{k+1}([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))$ is concave in y_1 and y_2 for given ϵ_1 and ϵ_2 ~~because expectation is a convex combination~~. Substituting (1) and (2) into $V_{k+1}([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))$ leads to

$$V_{k+1}([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2)) = V_{k+1}(((1-a_1)y_1 + b_1y_2 - \epsilon_1)^+, a_2y_1 + (1-b_2)y_2 - \epsilon_2).$$

Suppose that λ is a number between 0 and 1, y_1, y_2, y_1', y_2' are random numbers. We have

$$\begin{aligned}
&V_{k+1}([(1-a_1)(\lambda y_1 + (1-\lambda)y_1') + b_1(\lambda y_2 + (1-\lambda)y_2') - \epsilon_1]^+,\\
&a_2(\lambda y_1 + (1-\lambda)y_1') + (1-b_2)(\lambda y_2 + (1-\lambda)y_2') - \epsilon_2)\\
=&V_{k+1}([\lambda((1-a_1)y_1 + b_1y_2 - \epsilon_1) + (1-\lambda)((1-a_1)y_1' + b_1y_2' - \epsilon_2)]^+,\\
&\lambda(a_2y_1 + (1-b_2)y_2 - \epsilon_2) + (1-\lambda)(a_2y_1' + (1-b_2)y_2' - \epsilon_2))\\
\geq&V_{k+1}(\lambda((1-a_1)y_1 + b_1y_2 - \epsilon_1)^+ + (1-\lambda)((1-a_1)y_1' + b_1y_2' - \epsilon_1)^+,\\
&\lambda(a_2y_1 + (1-b_2)y_2 - \epsilon_2) + (1-\lambda)(a_2y_1' + (1-b_2)y_2' - \epsilon_2))\\
\geq&\lambda V_{k+1}(((1-a_1)y_1 + b_1y_2 - \epsilon_1)^+, a_2y_1 + (1-b_2)y_2 - \epsilon_2)\\
&+ (1-\lambda)V_{k+1}(((1-a_1)y_1' + b_1y_2' - \epsilon_1)^+, a_2y_1' + (1-b_2)y_2' - \epsilon_2).
\end{aligned}$$

The first inequality holds because $V_{k+1}(x_1, x_2)$ is decreasing in x_1. The second inequality holds because of the concavity of $V_{k+1}(x_1, x_2)$. Consequently, $EV_{k+1}([y_1 - D_1(y_1, y_2)]^+, y_2 - D_2(y_1, y_2))$ is concave in y_1 and y_2. Hence, all items in $H_k(y_1, y_2)$ are concave, and by ~~lemma~~ 1, we conclude that $V_k(x_1, x_2)$ is concave in x_1 and x_2. On the other hand, as x_1 ~~and~~ x_2

increase, the decision space $\{x_1 \le y_1 \le M_1, x_2^+ \le y_2 \le M_2\}$ diminishes. Thus, it is clear that $V_k(x_1, x_2)$ is decreasing in x_1 and x_2.

Proof of Lemma 2.

Denote $k_1 = r_1 + l_1 + h_1 - \gamma c_1$, $k_2 = (1-\gamma)r_2 + l_2 + h_2$, f_1 and f_2 are the probability density functions of ϵ_1 and ϵ_2 respectively.

$$
\begin{aligned}
\frac{\partial^2}{\partial y_1^2} C^+(y_1, y_2) = & -(1-a_1)^2 e_1 f_1((1-a_1)y_1 + b_1 y_2) - a_2^2 e_2 f_2(a_2 y_1 + (1-b_2)y_2) \le 0, \\
\frac{\partial^2}{\partial y_2^2} C^+(y_1, y_2) = & -b_1^2 k_1 f_1((1-a_1)y_1 + b_1 y_2) - (1-b_2)^2 k_2 f_2(a_2 y_1 + (1-b_2)y_2) \le 0, \\
\frac{\partial^2}{\partial y_1 \partial y_2} C^+(y_1, y_2) = C_{21}^+(y_1, y_2) = & -(1-a_1) b_1 k_1 f_1((1-a_1)y_1 + b_1 y_2) \\
& - a_2(1-b_2) k_2 f_2(a_2 y_1 + (1-b_2)y_2) \le 0.
\end{aligned} \tag{34}
$$

~~So~~,

$$
\begin{aligned}
& \frac{\partial^2}{\partial y_1^2} C^+(y_1, y_2) \frac{\partial^2}{\partial y_2^2} C^+(y_1, y_2) - (\frac{\partial^2}{\partial y_1 \partial y_2} C^+(y_1, y_2))^2 \\
= \; & (1 - a_1 - b_2 + a_1 b_2 - a_2 b_1)^2 k_1 k_2 f_1((1-a_1)y_1 + b_1 y_2) f_2(a_2 y_1 + (1-b_2)y_2)) \ge 0.
\end{aligned} \tag{35}
$$

Proof of Lemma 3.

We prove this lemma according to Ignall and Veinott (1969). We have obtained in the proof of Lemma 2 that

$$
\begin{aligned}
& \frac{\partial^2}{\partial y_1^2} C^+(y_1, y_2) \frac{\partial^2}{\partial y_2^2} C^+(y_1, y_2) - (\frac{\partial^2}{\partial y_1 \partial y_2} C^+(y_1, y_2))^2 \\
= \; & (1 - a_1 - b_2 + a_1 b_2 - a_2 b_1)^2 k_1 k_2 f_1((1-a_1)y_1 + b_1 y_2) f_2(a_2 y_1 + (1-b_2)y_2)) \ge 0.
\end{aligned} \tag{36}
$$

When we order inventories up to y_1 and y_2 for dual channels, $(1-a_1)y_1 + b_1 y_2 = y_1 - (a_1 y_1 - b_1 y_2)$ can be interpreted as the part used to satisfy demand of loyal customers of the physical store. So inequalities $v_1 \le (1-a_1)y_1 + b_1 y_2 \le u_1$ and $v_2 \le a_2 y_1 + (1-b_2)y_2) \le u_2$ hold in the optimal policy. Thus, we can restrict our decision space to $\{(y_1, y_2) : v_1 \le (1-a_1)y_1 + b_1 y_2 \le u_1, v_2 \le a_2 y_1 + (1-b_2)y_2) \le u_2, 0 \le y_1 \le M_1, 0 \le y_2 \le M_2\}$. Because F_1 and F_2 are strictly increasing, f_1 is larger than zero on $[v_1, u_1]$ and f_2 is larger than zero on $[v_2, u_2]$. As a result, as long as $1 - a_1 - b_2 + a_1 b_2 - a_2 b_1 \ne 0$, $C^+(y_1, y_2)$ is strictly concave on $\{(y_1, y_2) : v_1 \le (1-a_1)y_1 + b_1 y_2 \le u_1, v_2 \le a_2 y_1 + (1-b_2)y_2) \le u_2\}$. ~~So~~ y^* is unique.

Proof of Lemma 4. For simplicity, we denote $C_1^+(y_1, y_2)$ as taking derivative with the first component ~~that is y_1~~ of $C^+(y_1, y_2)$. $C_2^+(y_1, y_2)$, $C_{12}^+(y_1, y_2)$, $C_{21}^+(y_1, y_2)$, $C_{11}^+(y_1, y_2)$, $C_{22}^+(y_1, y_2)$ all have similar meanings.

According to the definition of $z_2(x_1)$, we have

$$
C_2^+(x_1, z_2(x_1)) = 0 \tag{37}
$$

Taking derivative of the two sides of equation (37) with respect to x_1, we have~~:~~ $C_{21}^+(x_1, z_2(x_1)) + z_2'(x_1) C_{22}^+(x_1, z_2(x_1)) = 0$. Because $C^+(y_1, y_2)$ is concave, ~~so~~ $C_{22}^+(x_1, z_2(x_1)) \le 0$ for arbitrary x and y. According to ~~lemma~~ 2, $C_{21}^+(x_1, z_2(x_1)) \le 0$, so $z_2'(x_1) \le 0$. Similarly, $z_1'(x_2) \le 0$.

Proof of Theorem 2.

~~We~~ have showed that as long as $1 - a_1 - b_2 + a_1 b_2 - a_2 b_1 \ne 0$, $C^+(y_1, y_2)$ is strictly concave on $\{(y_1, y_2) : v_1 \le (1-a_1)y_1 + b_1 y_2 \le u_1, v_2 \le a_2 y_1 + (1-b_2)y_2) \le u_2\}$ ~~in Lemma 3~~. So $\bar{y}(x_1, x_2)$ has the substitute property.

According to ~~lemma~~ 5 in Ignall and Veinott (1969), $\bar{y}(x_1, x_2)$ is the myopic policy. Refer to ~~lemma 5 and theorem 4~~ in Ignall and Veinott (1969), the myopic inventory policy is optimal ~~in~~ infinite horizon problem.

Proof of Proposition 1.

Take derivative of $y_1^{'}$ and $y_2^{'}$ with respect to a_1, b_1, a_2 and b_2 respectively.

$$\frac{\partial}{\partial a_1} y_1^{'} = \frac{A[\frac{(1-b_2)^2}{k_1}(F_1^{-1})'(D) + \frac{b_1^2}{k_2}(F_2^{-1})'(E)] + (1-b_2)C[(1-b_2)F_1^{-1}(D) - b_1 F_2^{-1}(E)]}{C^3}$$

$$\frac{\partial}{\partial a_1} y_2^{'} = \frac{-A[\frac{a_2(1-b_2)}{k_1}(F_1^{-1})'(D) + \frac{b_1(1-a_1)}{k_2}(F_2^{-1})'(E)] - a_2 C[(1-b_2)F_1^{-1}(D) - b_1 F_2^{-1}(E)]}{C^3}$$

$$\frac{\partial}{\partial b_2} y_1^{'} = \frac{-B[\frac{a_2(1-b_2)}{k_1}(F_1^{-1})'(D) + \frac{(1-a_1)b_1}{k_2}(F_2^{-1})'(E)] - b_1 C[(1-a_1)F_2^{-1}(E) - a_2 F_1^{-1}(D)]}{C^3}$$

$$\frac{\partial}{\partial b_2} y_2^{'} = \frac{B[\frac{(a_2)^2}{k_1}(F_1^{-1})'(D) + \frac{(1-a_1)^2}{k_2}(F_2^{-1})'(E)] + (1-a_1)C[(1-a_1)F_2^{-1}(E) - a_2 F_1^{-1}(D)]}{C^3}$$

where,

$$D = \frac{l_1}{k_1} + \frac{(r_1 - c_1)(1-b_2) - (r_2 - c_2)a_2}{[(1-a_1)(1-b_2) - a_2 b_1]k_1}$$

$$E = \frac{l_2 - \gamma(r_2 - c_2)}{k_2} - \frac{(r_1 - c_1)b_1 - (r_2 - c_2)(1-a_1)}{[(1-a_1)(1-b_2) - a_2 b_1]k_2}$$

Proof of Proposition 3.

~~Now, suppose~~ $y_2^* \le y_2^{''}$, we prove $y_1^* \ge y_1^{''}$. Denote

$$E_1 = \frac{l_1}{k_1} + \frac{(r_1 - c_1)(1-b_2) - (r_2 - c_2)a_2}{[(1-a_1)(1-b_2) - a_2 b_1]k_1}$$

$$E_2 = \frac{l_2 - \gamma(r_2 - c_2)}{k_2} - \frac{(r_1 - c_1)b_1 - (r_2 - c_2)(1-a_1)}{[(1-a_1)(1-b_2) - a_2 b_1]k_2}$$

~~So~~

$$y_1^* = \frac{(1-b_2)F_1^{-1}(E_1) - b_1 F_2^{-1}(E_2)}{(1-a_1)(1-b_2) - a_2 b_1},$$

$$y_2^* = \frac{(1-a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)}{(1-a_1)(1-b_2) - a_2 b_1}.$$

$$y_2^* - y_2^{''} = \frac{(1-a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)}{(1-a_1)(1-b_2) - a_2 b_1} - F_2^{-1}(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2})$$

$$= \frac{(1-a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1) - [(1-a_1)(1-b_2) - a_2 b_1]F_2^{-1}(\frac{(1-\gamma)(r_2-c_2)+l_2}{k_2})}{(1-a_1)(1-b_2) - a_2 b_1}$$

Because $(1-a_1)(1-b_2) - a_2 b_1 \ge 0$,

$$(1-a_1)F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1) - [(1-a_1)(1-b_2) - a_2 b_1]F_2^{-1}(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2}) \le 0$$

Because $D \ge 0$, so $E_2 - \frac{(1-\gamma)(r_2-c_2)+l_2}{k_2} \ge 0$. As F_2^{-1} is increasing, $F_2^{-1}(E_2) - F_2^{-1}(\frac{(1-\gamma)(r_2-c_2)+l_2}{k_2}) \ge 0$. ~~So~~

$$[(1-a_1)b_2 + a_2 b_1]F_2^{-1}(E_2) - a_2 F_1^{-1}(E_1)$$

$$\le (1-a_1)F_2^{-1}(E_2) - [(1-a_1)(1-b_2) - a_2 b_1]F_2^{-1}(\frac{(1-\gamma)(r_2 - c_2) + l_2}{k_2}) - a_2 F_1^{-1}(E_1)$$

$$\le 0$$

Now, we prove $y_1^{*} - y_1^{''} \geq 0$.

$$y_1^{*} - y_1^{''} = \frac{(1-b_2)F_1^{-1}(E_1) - b_1F_2^{-1}(E_2)}{(1-a_1)(1-b_2)-a_2b_1} - F_1^{-1}(\frac{r_1-c_1+l_1}{k_1})$$
$$= \frac{(1-b_2)F_1^{-1}(E_1) - b_1F_2^{-1}(E_2) - [(1-a_1)(1-b_2)-a_2b_1]F_1^{-1}(\frac{r_1-c_1+l_1}{k_1})}{(1-a_1)(1-b_2)-a_2b_1}$$

Because $C \geq 0$, $F_1^{-1}(E_1) \geq F_1^{-1}(\frac{r_1-c_1+l_1}{k_1})$. So, $y_1^{*} - y_1^{''} \geq \frac{[(1-b_2)a_1+a_2b_1]F_1^{-1}(E_1)-b_1F_2^{-1}(E_2)}{(1-a_1)(1-b_2)-a_2b_1}$.
~~In which,~~

$$\begin{aligned}
&[(1-b_2)a_1+a_2b_1]F_1^{-1}(E_1) - b_1F_2^{-1}(E_2) \\
&= -\frac{b_1}{(1-a_1)b_2+a_2b_1}[((1-a_1)b_2+a_2b_1)F_2^{-1}(E_2) - a_2F_1^{-1}(E_1)] \\
&+ [((1-b_2)a_1+a_2b_1) - \frac{a_2b_1}{(1-a_1)b_2+a_2b_1}]F_1^{-1}(E_1) \\
&= -\frac{b_1}{(1-a_1)b_2+a_2b_1}[((1-a_1)b_2+a_2b_1)F_2^{-1}(E_2) - a_2F_1^{-1}(E_1)] \\
&+ \frac{[(1-a_1)b_2+a_2b_1][(1-b_2)a_1+a_2b_1]-a_2b_1}{(1-a_1)b_2+a_2b_1}F_1^{-1}(E_1)
\end{aligned}$$

The first item is larger than zero. Now, we need to show the second item is larger than zero. It suffices to show that $[(1-a_1)b_2+a_2b_1][(1-b_2)a_1+a_2b_1]-a_2b_1$ is larger than zero.

$$\begin{aligned}
&[(1-a_1)b_2+a_2b_1][(1-b_2)a_1+a_2b_1]-a_2b_1 \\
&= (1-a_1)(1-b_2)a_1b_2 + (a_2b_1)^2 + (1-a_1)a_2b_1b_2 + (1-b_2)a_1a_2b_1 - a_2b_1 \\
&= [(1-a_1)b_2 + (1-b_2)a_1]a_2b_1 - (1-a_2b_1)a_2b_1 + (1-a_1)(1-b_2)a_1b_2 \\
&= a_2b_1(a_2b_1 - a_1b_2) + (1-a_1)(1-b_2)(a_1b_2 - a_2b_1) \\
&= [(1-a_1)(1-b_2)-a_2b_1](a_1b_2 - a_2b_1)
\end{aligned} \tag{38}$$

In which,

$$(1-a_1)(1-b_2)-a_2b_1 = (1-a_1-b_2) + (a_1b_2 - a_2b_1) \geq 0 \tag{39}$$

The above inequality holds because $a_1 + b_2 \leq 1$, $a_1 \geq b_1$ and $b_2 \geq a_2$. Until now, we have proved $y_1^{*} \geq y_1^{''}$. Similarly, when $y_1^{*} \leq y_1^{''}$ $y_2^{*} \geq y_1^{''}$