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The Price of Fairness with the Extended Perles-Maschler Solution

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Abstract In Nash bargaining problem, due to fairness concern of players, instead of maximizing the sum of utilities of all players, an implementable solution should satisfy some axioms or characterizations. Such a solution can result in the so-called price of fairness, because of the reduction in the sum of utilities of all players. An important issue is to quantify the system efficiency loss under axiomatic solutions through the price of fairness. Based on Perles-Maschler solution of two-player Nash bargaining problem, this paper deals with the extended Perles-Maschler solution of multi-player Nash bargaining problem. We give lower bounds of three measures of the system efficiency for this solution, and show that the lower bounds are asymptotically tight.

Keywords Bargaining problem · Perles-Maschler solution · Price of fairness · Convexity · Matrices

1 Introduction

In general, Nash bargaining problem involves a group of players who have the opportunity to collaborate for mutual benefit (Nash 1950 and Roth 1979). One of common objectives is to maximize the sum of utilities of all players. However, the corresponding system optimal solution may not be implementable

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due to possible “unfair” to some of players for the expense to achieve such a solution (Bertsimas et al. 2011). Consequently, it is important to have a solution being implementable based on particular characterizations which are acceptable by all players. These characterizations lead to several axioms, some of which are directly or indirectly relevant to “fairness” concern. Usually, an axiomatic solution deviates from the system optimal solution. The resultant relative system efficiency loss (i.e., relative difference of the sum of utilities of all players to the maximum system utility) due to the solution deviation is called price of fairness (POF) (Bertsimas et al. 2011).

To the best of our knowledge, the first axiomatic solution of Nash bargaining problem is the Nash solution (see Nash 1950), which satisfies four axioms. Later, Kalai and Smorodinsky (1975) propose a different axiomatic solution (with four axioms), with three axioms being the same as the Nash solution. (We call it KS solution.) Furthermore, Perles and Maschler (1981) propose a new axiomatic solution (with five axioms), with three axioms being the same as the Nash solution. (We hereafter call it PM solution.) The key different point between the PM solution and others is that the PM solution satisfies an important axiom of superadditivity: aggregating two Nash bargaining problems into one can benefit both players. Salonen (1985) proposes another solution (with four axioms) which is monotonic with respect to coin tossing method. It is noted that the above four solutions are solutions of two-player Nash bargaining problem.

It is shown that the Nash solution and the KS solution can be generalized to multi-player Nash bargaining problem without changing their axioms (Roth 1979 and Imai 1983). And another characterization of the KS solution is given by Chang and Hwang (1999). Later, Hinojosa et al. (2005) generalize the KS solution to the case when each player has multiple criteria to value his outcome. However, the extension of the PM solution without changing the axioms is proved to be impossible (Perles 1982). Nevertheless, with an geometric procedure described by Perles and Maschler (1981), instead of axioms, Calvo and Gutierrez (1994) extend the PM solution to multi-player bargaining problem and prove that this geometric procedure is equivalent to the axioms in the PM solution of two-player Nash bargaining problem. (We denote this solution by extended PM solution, or EPM solution for short.)

For multi-player Nash bargaining problem, the paper of Bertsimas et al. (2011) is the first one to analyze the POFs of the Nash solution and the KS solution. The authors present tight lower bounds of the POF for both solutions which depend on a single parameter-the number of players. Recently, Bertsimas et al. (2012) extend the results to a family of solutions parameterized by a single parameter that measures the aversion to inequality. In particular, they provide near-tight upper bounds on the relative efficiency loss compared to the system optimal solution, as well as tight upper bounds on the relative fairness loss where fairness is measured by the minimum utility of players.

On the other hand, the PM solution of two-player Nash bargaining problem is also well accepted and widely studied (see, e.g., Thomson 1994 and Pallaschke and Rosenmuller 2007). Although the EPM solution is not characterized by axioms, the geometric procedure proposed by Calvo and Gutierrez (1994) characterizes the bargaining process of all players. That is, each player initially demands his highest possible payoff, if this outcome is achievable, the bargaining ends with this outcome. Otherwise, each player decreases his initial payoff continuously until they reach an agreement (i.e. an achievable outcome). This characterization does not only gives a result of the bargaining problem but also provides a reasonable bargaining process. Hence in this paper, we aim to characterize the POFs of three relevant measures of the EPM solution (for multi-player Nash bargaining problem).

The structure of this paper is organized as follows. In Section 2, we introduce the notation of Nash bargaining problem and the geometric procedure of the EPM solution. More detailed characterizations of the EPM solution are introduced in Section 3. The main result of this paper is presented in Section 4. In Section 5, examples are given to demonstrate our bounds being asymptotically tight. The paper is concluded in Section 6.

2 Nash Bargaining Problem and Geometric Procedure of the EPM Solution

In this section, we briefly introduce Nash bargaining problem and alternative characterizations of the EPM solution proposed by Calvo and Gutierrez (1994). To begin with, we define the following notation.

$\mathbf{0}^n = (0, \dots, 0)^T \in \mathbb{R}^n$;

$\mathbf{e} = (1, \dots, 1)^T \in \mathbb{R}^n$;

$\mathbf{e}^i = (0, \dots, 0, 1, 0 \dots, 0)^T \in \mathbb{R}^n$ — the ith coordinate is 1, for $i = 1, 2, \dots, n$.

For $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $\mathbf{y} = (y_1, \dots, y_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $A \subset \mathbb{R}^n$, we define the following items:

1) an order is defined as: $\mathbf{x} \geq (>) \mathbf{y} \iff x_i \geq (>) y_i$ for $i = 1, 2, \dots, n$;

2) the following vectors are defined: $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)^T, i = 1, 2, \dots, n$;

$(\mathbf{x}_{-i} : u) = (x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n)^T, i = 1, 2, \dots, n$;

3) the following sets are defined: $OP(A) = \{\mathbf{z} \in A |$ if $\mathbf{w} \geq \mathbf{z}$ and $\mathbf{w} \in A$,then $\mathbf{w} = \mathbf{z}\}$;

$A^i = \{\mathbf{z}_{-i} | \mathbf{z} \in A\}$;

4) the following functions are defined: $g^i(\mathbf{x}_{-i}|A) = \sup\{u \in \mathbb{R} | (\mathbf{x}_{-i} : u) \in A\}, \forall \mathbf{x} \in A, i = 1, 2, \dots, n$. Then $\forall \mathbf{x} \in A$, we have $g^i(\mathbf{x}_{-i}|A) \geq x_i, i = 1, 2, \dots, n$. Furthermore, define $\mathbf{g}(\mathbf{x}|A) = (g^1(\mathbf{x}_{-1}|A), \dots, g^n(\mathbf{x}_{-n}|A))^T, \forall \mathbf{x} \in A$.

In the following, we formulate Nash bargaining problem.

Definition 1. A pair $(\mathbf{a}, A)$ is a bargaining problem if A is a compact and convex subset of $\mathbb{R}^n$, and $\mathbf{a} \in \mathbb{R}^n$. Let U be the collection of all these pairs. (A is called a bargaining set, and $\mathbf{a}$ is called the disagreement point.)

Definition 2. A bargaining solution is a function $\mathbf{f}$ defined on U such that $\mathbf{f}$ satisfies $\mathbf{a} \leq \mathbf{f}(\mathbf{a}, A) \in A$.

Since the EPM solution $\mathbf{f_{PM}}$ has the property $\mathbf{f_{PM}}(\mathbf{a}, A) = \mathbf{f_{PM}}(\mathbf{0}^n, A - \mathbf{a}) + \mathbf{a}$, without loss of generality we assume the disagreement point $\mathbf{a}$ to be the origin, i.e., $\mathbf{a} = \mathbf{0}^n$, and also we assume $A \subset \mathbb{R}^n_+$. Thus, we can write A instead of $(\mathbf{0}^n, A)$ for short, i.e., a bargaining set A characterizes a Nash bargaining problem. According to Calvo and Gutierrez (1994), the EPM solution is originally defined on a subset of U, which consists of the bargaining sets satisfying the following three assumptions:

Assumption 1. $A \subset \mathbb{R}^n_+$ is convex, compact, comprehensive and $\exists \mathbf{a} \in A$ such that $\mathbf{a} > \mathbf{0}^n$, where comprehensive means if $\mathbf{u} \in A$ and $\mathbf{0}^n \leq \mathbf{v} \leq \mathbf{u}$, then $\mathbf{v} \in A$.

Assumption 2. For each $i = 1, 2, \ldots, n, g^i(\cdot|A)$ is continuous and strictly decreasing, and $\forall i = 1, 2, \ldots, n$, $\mathbf{x} \in OP(A) \Longleftrightarrow x_i = g^i(\mathbf{x}_{-i}|A)$.

Assumption 3. $\forall i = 1, 2, \ldots, n, g^i(\cdot|A) \in C^2(A^i)$.

Let B^n_+ be the collection of all sets satisfying Assumptions 1, 2 and 3. Then, B^n_+ is the subset of U. In analysis hereafter, we are interested in B^n_+ instead of U. Nevertheless, it is known from Calvo and Gutierrez (1994) that there exists a unique EPM solution defined on B^n_+ .

Given $A \in B^n_+$, Calvo and Gutierrez (1994) define a so-called PM path $C(A)$ as follows.

Definition 3. Given $A \in B^n_+$, the parametric equation of the PM path $C(A)$ is $\eta(x_1; A) = (x_1, \eta_2(x_1|A), \ldots, \eta_n(x_1|A))^T$,$x_1 \in [0, g^1(\mathbf{0}^{n-1}|A)]$ if, $\forall i = 2, \ldots, n, \eta_i(\cdot|A)$ is a continuous, strictly increasing and differentiable function satisfying,

i) $\eta_i(0|A) = 0, \forall i = 2, \ldots, n$.

ii) $\forall \mathbf{x} \in C(A) \cap A$,

ii.1) $\mathbf{g}(\mathbf{x}|A) = (g^1(\mathbf{x}_{-i}|A), \ldots, g^n(\mathbf{x}_{-n}|A))^T \in C(A)$.

ii.2) $\eta'_i(x_1|A) = \eta'_i(g^1(\mathbf{x}_{-i}|A)|A), \forall i = 2, \ldots, n$.

Then, the EPM solution $\mathbf{f_{PM}}(A)$ given by Calvo and Gutierrez (1994) is the intersection of the $C(A)$ and $OP(A)$, i.e., $\mathbf{f_{PM}}(A) \in C(A) \cap OP(A)$.

3 The EPM Solution

In this section, we briefly introduce the characterization of the EPM solution proposed by Calvo and Gutierrez (1994). They have proved that the tangent vector of $C(A)$ must satisfy an eigenvalue and eigenvector equation. Additionally, they have proved that the curve $C(A)$ is determined uniquely by some ordinary differential equations.

3.1 The eigenvalue and eigenvector equation

For $\forall A \in B_+^n$ and $\mathbf{x} \in A$, define a matrix $\mathbf{G}(\mathbf{x}|A) = \partial \mathbf{g}(\mathbf{x}|A)/\partial \mathbf{x}$, i.e.,

$$\mathbf{G}(\mathbf{x}|A) = \begin{bmatrix} 0 & g_2^1(\mathbf{x}_{-1}|A) & \cdots & g_{n-1}^1(\mathbf{x}_{-1}|A) & g_n^1(\mathbf{x}_{-1}|A) \\ g_1^2(\mathbf{x}_{-2}|A) & 0 & \cdots & g_{n-1}^2(\mathbf{x}_{-2}|A) & g_n^2(\mathbf{x}_{-2}|A) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_1^{n-1}(\mathbf{x}_{-n+1}|A) & g_2^{n-1}(\mathbf{x}_{-n+1}|A) & \cdots & 0 & g_n^{n-1}(\mathbf{x}_{-n+1}|A) \\ g_1^n(\mathbf{x}_{-n}|A) & g_2^n(\mathbf{x}_{-n}|A) & \cdots & g_{n-1}^n(\mathbf{x}_{-n}|A) & 0 \end{bmatrix}.$$

Here $g_i^j(\mathbf{x}_{-j}|A) = \partial g^j(\mathbf{x}_{-j}|A)/\partial x_i$ for $i \neq j$. Then, $\forall \mathbf{x} \in A$, consider the following eigenvalue and eigenvector equation,

$$\lambda(\mathbf{x}|A) \cdot \mathbf{h}(\mathbf{x}|A) = \mathbf{G}(\mathbf{x}|A) \cdot \mathbf{h}(\mathbf{x}|A) \quad s.t \quad \mathbf{h}(\mathbf{x}|A) \geq 0 \quad , \quad \lambda(\mathbf{x}|A) \leq 0. \quad (1)$$

Calvo and Gutierrez (1994) show that, $\forall \mathbf{x} \in A$, equation (1) has a unique solution of the eigenvalue $\lambda(\mathbf{x}|A)$ and a unique associated eigenvector $\mathbf{h}(\mathbf{x}|A)$ satisfying $\lambda(\mathbf{x}|A) < 0$ and $\mathbf{h}(\mathbf{x}|A) = (1, h_2(\mathbf{x}|A), \ldots, h_n(\mathbf{x}|A)) > 0$.

3.2 The ordinary differential equations

The ordinary differential equations that uniquely determine the parametric equation of the PM path C(A) are,

$$\begin{cases} \eta'(x_1|A) = \mathbf{h}(\eta(x_1|A)|A), \\ \eta(0|A) = \mathbf{0}^n. \end{cases} \quad (2)$$

The curve $C(A)$ is formed in the following manner. It consists of two parts: a curve $C_1(A)$ in A and a curve $C_2(A)$ outside A. The curve $C_1(A) = C(A) \cap A$ starts from $\mathbf{0}^n$ and moves along the direction $\mathbf{h}(\mathbf{x}|A)$ until reaching the bound of A, i.e., reaching the point $\mathbf{f_{PM}}(A)$. The rest part $C_2(A)$ is determined by $C_2(A) = \mathbf{g}(C_1(A)|A)$. Then, $C(A) = C_1(A) \bigcup C_2(A)$.

4 The Efficiency of the EPM solution

In this section, we analyze the system efficiency loss of the EPM solution. For expositional convenience, throughout the paper we denote $\mathbf{f_{PM}}(A) = (f^1(A), \ldots, f^n(A))$. Three measures that characterize the system efficiency are considered. The first one is defined in Bertsimas et al. (2011), which is not scale invariant:

$$E_A^e = \frac{\mathbf{e}^T \cdot \mathbf{f_{PM}}(A)}{\max_{\mathbf{x} \in A} \mathbf{e}^T \cdot \mathbf{x}} = \frac{\sum_{i=1}^n f^i(A)}{\max_{\mathbf{x} \in A} \sum_{i=1}^n x_i}. \quad (3)$$

Other two measures are both scale invariant:

$$E_A^g = \frac{\sum_{i=1}^n [g^i(\mathbf{0^{n-1}}|A)]^{-1} \cdot f^i(A)}{\max_{\mathbf{x}\in A} \sum_{i=1}^n [g^i(\mathbf{0^{n-1}}|A)]^{-1} \cdot x_i} \tag{4}$$

and

$$E_A^f = \frac{n}{\max_{\mathbf{x}\in A} \sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i}. \tag{5}$$

Similarly to Bertsimas et al. (2011), we define the POFs of the EPM solution to be $1 - E_A^e$, $1 - E_A^g$ and $1 - E_A^f$ respectively.

Our goal is to give lower bounds of the ratios E_A^e, E_A^g and E_A^f, which are proved to be asymptotically tight in Section 5. To do this, we divide the proof into three steps: in the first step, we show that the absolute value of $\lambda(\mathbf{x}|A)$ has a upper bound (i.e., Theorem 1); in the second step, we see that in the EPM solution, the utility of each player has a lower bound (i.e., Theorem 2); and in the third step, we use the lower bound in the second step to estimate E_A^e, E_A^g and E_A^f. (i.e., Theorem 3).

Theorem 1 *$\forall A \in B_+^n, \mathbf{x} = (x_1, \ldots, x_n)^T \in A$ and $\mathbf{x} \notin OP(A)$, the eigenvalue $\lambda(\mathbf{x}|A)$ of the matrix $\mathbf{G}(\mathbf{x}|A)$ satisfies $|\lambda(\mathbf{x}|A)| \leq n - 1$.*

Proof : To prove this theorem, we consider another matrix $\mathbf{V}$ which has the same eigenvalues as $\mathbf{G}(\mathbf{x}|A)$, and then prove that the elements of $\mathbf{V}$ are in the interval $[-1, 0]$.

Since $\mathbf{x} \in A$ and $\mathbf{x} \notin OP(A)$, we have $g^i(\mathbf{x}_{-i}|A) > x_i$ for $i = 1, \ldots, n$. Define the following matrix

$$\mathbf{J} = diag\{g^1(\mathbf{x}_{-1}|A) - x_1, \ldots, g^n(\mathbf{x}_{-n}|A) - x_n\}. \tag{6}$$

Let $\mathbf{V} = \{v_{ij}\}_{i,j=1}^n = \mathbf{J}^{-1}\mathbf{G}(\mathbf{x}|A)\mathbf{J}$. Then, $\mathbf{V}$ has the same eigenvalues as $\mathbf{G}(\mathbf{x}|A)$.

Since $\mathbf{V} = \mathbf{J}^{-1}\mathbf{G}(\mathbf{x}|A)\mathbf{J}$, we have,

$$v_{ii} = \frac{g^i(\mathbf{x}_{-i}|A) - x_i}{g^i(\mathbf{x}_{-i}|A) - x_i} \cdot 0 = 0. \tag{7}$$

This implies that the diagonal elements of $\mathbf{V}$ are all zeros.

$\forall \alpha, \beta \in A^k$ and $u \in [0, 1]$, define $\alpha_{+k} = (\alpha_1, \ldots \alpha_{k-1}, g^k(\alpha|A), \alpha_k, \ldots, \alpha_{n-1})$, $\beta_{+k} = (\beta_1, \ldots \beta_{k-1}, g^k(\beta|A), \beta_k, \ldots, \beta_{n-1})$ and $\gamma = u\alpha_{+k} + (1 - u)\beta_{+k}$. According to the definition of $g^k(\cdot|A)$, together with A being compact, we have $\alpha_{+k} \in A$ and $\beta_{+k} \in A$. Since A is convex, we have $\gamma = u\alpha_{+k} + (1-u)\beta_{+k} \in A$. From the definition of $g^k(\cdot|A)$, it holds that

$$\begin{aligned} g^k(u\alpha + (1-u)\beta|A) = g^k((u\alpha_{+k} + (1-u)\beta_{+k})_{-k}|A) = g^k(\gamma_{-k}|A) \\ \geq \gamma_k = ug^k(\alpha|A) + (1-u)g^k(\beta|A). \end{aligned} \tag{8}$$

Hence, $\forall 1 \leq k \leq n$, $g^k(\cdot|A)$ is concave in A^k.

For $i \neq j$,

$$v_{ij} = \frac{g^j(\mathbf{x}_{-j}|A) - x_j}{g^i(\mathbf{x}_{-i}|A) - x_i} g^i_j(\mathbf{x}_{-i}). \tag{9}$$

Since $\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x} = (\mathbf{x}_{-j} : g^j(\mathbf{x}_{-j}|A)) \in OP(A) \subseteq A$ (see the definition of $g^i(\cdot|A)$), we have $(\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_{-i} \in A^i$. Because $g^i(\cdot|A)$ is concave in A^i and $\mathbf{x}_{-i} \in A^i$, we have,

$$\begin{aligned} g^i_j(\mathbf{x}_{-i}|A) \geq & \frac{g^i(\mathbf{x}_{-i}|A) - g^i((\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_{-i}|A)}{x_j - (\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_j} \\ = & \frac{g^i(\mathbf{x}_{-i}|A) - g^i((\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_{-i}|A)}{x_j - g^j(\mathbf{x}_{-j}|A)}. \end{aligned} \tag{10}$$

Therefore, by (9),

$$v_{ij} \geq -\frac{g^i(\mathbf{x}_{-i}|A) - g^i((\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_{-i}|A)}{g^i(\mathbf{x}_{-i}|A) - x_i}. \tag{11}$$

Since $\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x} = (\mathbf{x}_{-j} : g^j(\mathbf{x}_{-j}|A)) \in OP(A)$, we have from Assumption 2 that $g^i((\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_{-i}|A) = (\mathbf{J} \cdot \mathbf{e}^j + \mathbf{x})_i = x_i$. Hence,

$$v_{ij} \geq -1. \tag{12}$$

Also according to Assumption 2, $g^i(\cdot|A)$ is decreasing, thus (9) leads to

$$v_{ij} \leq 0. \tag{13}$$

From (12) and (13), it is known that the elements of $\mathbf{V}$ are in the interval $[-1, 0]$.

Now we have proved that the elements of $\mathbf{V}$ are in the interval $[-1, 0]$ and the diagonal elements of $\mathbf{V}$ are all zero. According to Gershgorin circle theorem (see Varga (2009)), the norms of all the eigenvalues of $\mathbf{V}$ are no greater than $n-1$. Since $\lambda(\mathbf{x}|A)$ is an eigenvalue of $\mathbf{G}(\mathbf{x}|A)$, and $\mathbf{G}(\mathbf{x}|A)$ and $\mathbf{V}$ have the same eigenvalues, we have $|\lambda(\mathbf{x}|A)| \leq n-1.\square$

The next theorem is a generalization of a result in Salonen (1985).

Theorem 2 $\forall A \in B^n_+$, $\mathbf{f_{PM}}(A) \geq \mathbf{g}(\mathbf{0}^n|A)/n$.

Proof : To prove the inequality, here we construct a function $\mathbf{W}(u) : \mathbb{R} \longmapsto \mathbb{R}^n$ in that $\exists u_0 \geq 0$ it follows $\mathbf{W}(u_0) \geq \mathbf{0^n} \iff \mathbf{f_{PM}}(A) \geq g(\mathbf{0}^n|A)/n$. Therefore, proving $\mathbf{W}(0) = \mathbf{0}^n$ and $\mathbf{W}'(u) \geq \mathbf{0}^n$ for $0 \leq u < u_0$ is sufficient.

Consider a function

$$\mathbf{W}(u|A) = (n-1)\eta(u|A) + \mathbf{g}(\eta(u|A)|A) - \mathbf{g}(\mathbf{0}^n|A). \tag{14}$$

Taking the derivative of $\mathbf{W}(u|A)$ with respect to u and applying (1) and (2), we have

$$\begin{aligned} \mathbf{W}'(u|A) = & (n-1)\eta'(u|A) + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\eta(u|A)|A) \cdot \eta'(u|A) \\ = & (n-1+\lambda(\eta(u|A)|A)) \cdot \mathbf{h}(\eta(u|A)|A). \end{aligned} \tag{15}$$

Assume $\mathbf{f_{PM}}(A) = \eta(u_{PM}|A) \in OP(A)$. Now we consider $0 \le u < u_{PM}$. Since η is strictly increasing by Definition 3, we have $\eta(u|A) \in A/OP(A)$. According to Theorem 1 we know that $|\lambda(\eta(u|A)|A)| \le n-1$. According to (1), we also know that $\mathbf{h}(\eta(u|A)|A) \ge \mathbf{0}^n$. Therefore, by (15), we have $\mathbf{W}'(u|A) \ge \mathbf{f_{PM}}(A)$, which implies $W(u_{PM}|A) \ge W(0|A) = \mathbf{0}^n$, i.e.,

$$\mathbf{W}(u_{PM}|A) = (n-1)\eta(u_{PM}|A) + \mathbf{g}(\eta(u_{PM}|A)|A) - \mathbf{g}(\mathbf{0}^n|A) \ge \mathbf{0}^n. \quad (16)$$

Since $\eta(u_{PM}|A) = \mathbf{f_{PM}}(A) \in OP(A)$, according to Assumption 2, we have $\eta(u_{PM}|A) = \mathbf{g}(\eta(u_{PM}|A)|A)$. Thus, $\mathbf{W}(u_{PM}|A) \ge \mathbf{0^n} \Longleftrightarrow \mathbf{f_{PM}}(A) \ge g(\mathbf{0}^n|A)/n$, where u_{PM} can be regarded as u_0. By (16), we have $\mathbf{f_{PM}}(A) = \eta(u_{PM}|A) \ge \mathbf{g}(\mathbf{0}^n|A)/n$, which completes the proof.□

With the help of the above two theorems, we now estimate E_A^e, E_A^g and E_A^f.

Theorem 3 $\forall A \in B_+^n$,

$$E_A^e \ge \frac{1}{n} + \frac{n-1}{n} \cdot \frac{\min_{1\le i\le n} g^i(\mathbf{0}^{n-1}|A)}{\Sigma_{i=1}^n g^i(\mathbf{0}^{n-1}|A)}, \quad (17)$$

$$E_A^g \ge \frac{2n-1}{n^2} \quad (18)$$

and

$$E_A^f \ge \frac{n}{n^2-n+1}. \quad (19)$$

Before providing the proof of the theorem, we give the following sketch idea of the procedure to estimate E_A^e. (E_A^g can be easily estimated by using the estimation of E_A^e, and E_A^f can be easily estimated directly.)

Since A is compact, there exists $\mathbf{z} = (z_1, \ldots, z_n)^T \in A$ such that $\mathbf{e}^T \cdot \mathbf{z} = \max_{\mathbf{x}\in A} \mathbf{e}^T \cdot \mathbf{x}$. It is clear that $\mathbf{z} \in OP(A)$. We define the following linear function:

$L(\mathbf{x}) = \sum_{i=1}^n (x_i/g^i(\mathbf{0}^{n-1}|A))$;

and further define the following sets:

$S = \{\mathbf{x}|\mathbf{x} \ge \mathbf{0}^n, L(\mathbf{x}) \le 1\}$;

$\tilde{S} = \{\mathbf{x}|\mathbf{x} \ge \mathbf{0}^n, L(\mathbf{x}) = 1, \exists j, x_j = 0\}$;

$D = \{b\mathbf{z} + (1-b)\mathbf{x}|b \in [0,1], \mathbf{x} \in \tilde{S}\}$;

$\tilde{D} = \{\mathbf{x} \in D|\mathbf{x} \ge \mathbf{g}(\mathbf{0}^n|A)/n\}$;

$\tilde{A} = \{\mathbf{x} \in A|\mathbf{x} \ge \mathbf{g}(\mathbf{0}^n|A)/n\}$.

Figure 1 is helpful for understanding these sets. From the figure, we have an insight of estimation of E_A^e for $n = 2$ as follow.

$$E_A^e = \frac{\mathbf{e}^T \cdot \mathbf{f_{PM}}(A)}{\mathbf{e}^T \cdot \mathbf{z}} \ge \frac{\mathbf{e}^T \cdot \mathbf{b}}{\mathbf{e}^T \cdot \mathbf{z}}. \quad (20)$$

Since $\mathbf{e}^T \cdot \mathbf{z} \ge \mathbf{e}^T \cdot \mathbf{d}$, we know that

$$E_A^e = \frac{\mathbf{e}^T \cdot \mathbf{f_{PM}}(A)}{\mathbf{e}^T \cdot \mathbf{z}} \ge \frac{\mathbf{e}^T \cdot \mathbf{b}}{\mathbf{e}^T \cdot \mathbf{z}} \ge \frac{\mathbf{e}^T \cdot \mathbf{a}}{\mathbf{e}^T \cdot \mathbf{z}}. \quad (21)$$

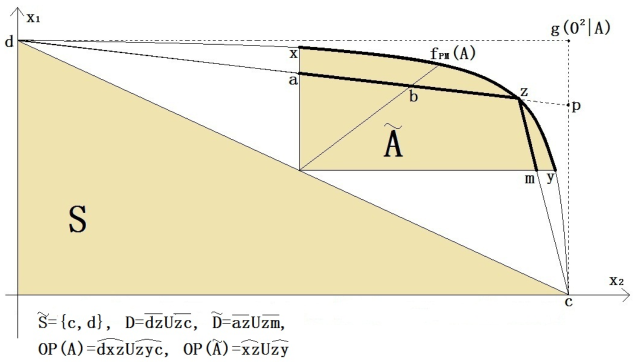

Fig. 1 A two dimensional example of Theorem 3.

Write $\mathbf{a}$ as a linear combination of $\mathbf{z}$ and $\mathbf{d}$, then it follows that

$$\begin{aligned}
E_A^e &\geq \frac{\mathbf{e}^T \cdot \mathbf{a}}{\mathbf{e}^T \cdot \mathbf{z}} = \frac{|\mathbf{da}|}{|\mathbf{dz}|} + \left(1 - \frac{|\mathbf{da}|}{|\mathbf{dz}|}\right) \cdot \frac{\mathbf{e}^T \cdot \mathbf{d}}{\mathbf{e}^T \cdot \mathbf{z}} \\
&\geq \frac{|\mathbf{da}|}{|\mathbf{dz}|} + \left(1 - \frac{|\mathbf{da}|}{|\mathbf{dz}|}\right) \cdot \frac{\mathbf{e}^T \cdot \mathbf{d}}{\mathbf{e}^T \cdot \mathbf{g}(\mathbf{0}^2|A)} \\
&= \frac{|\mathbf{da}|}{|\mathbf{dz}|} + \left(1 - \frac{|\mathbf{da}|}{|\mathbf{dz}|}\right) \cdot \frac{g^1(0|A)}{g^1(0|A) + g^2(0|A)} \\
&\geq \frac{|\mathbf{da}|}{|\mathbf{dp}|} + \left(1 - \frac{|\mathbf{da}|}{|\mathbf{dp}|}\right) \cdot \frac{g^1(0|A)}{g^1(0|A) + g^2(0|A)} \\
&= \frac{1}{2} + \frac{1}{2} \cdot \frac{g^1(0|A)}{g^1(0|A) + g^2(0|A)} \geq \frac{1}{2} + \frac{1}{2} \cdot \frac{\min\{g^1(0|A), g^2(0|A)\}}{g^1(0|A) + g^2(0|A)}.
\end{aligned}$$

Hence, the lower bound of E_A^e in Theorem 3 for $n = 2$ is obtained.

The above gives us an idea of how to estimate the lower bound of E_A^e.

Proof : We first estimate E_A^e.

It is known that S is the convex hull of $\{\mathbf{0}^n, g^i(\mathbf{0}^{n-1}|A)\mathbf{e}^i, i = 1, \dots n\}$. From the definition of $g^i(\cdot|A)$, together with A being compact, we know that $g^i(\mathbf{0}^{n-1}|A)\mathbf{e}^i = (\mathbf{0}^n_{-i} : g^i(\mathbf{0}^n_{-i}|A)) \in A, i = 1, \dots n$. Since A is convex and comprehensive, it is known that $S \subseteq A$. Note that $\mathbf{z} \in OP(A)$, hence with $S \subseteq A$, it follows that $\mathbf{z} \in OP(S)$ or $\mathbf{z} \notin S$.

According to Theorem 2, we know that $\mathbf{f_{PM}}(A) \geq \mathbf{g}(\mathbf{0}^n|A)/n$, which implies $\mathbf{f_{PM}}(A) \in OP(\tilde{A})$. Hence,

$$\mathbf{e}^T \cdot \mathbf{f_{PM}}(A) \geq \min_{\mathbf{x} \in OP(\tilde{A})} \mathbf{e}^T \cdot \mathbf{x}. \tag{22}$$

For any given $\mathbf{x} \in OP(\tilde{A})$, we then show that there exists a $\tilde{\mathbf{x}} \in D$ such that $\tilde{\mathbf{x}}$ is a point on the segment ϕ, where the end points of ϕ are $\mathbf{x}$ and $\mathbf{g}(\mathbf{0}^n|A)/n$. From the definition of D, it is seen that D separates $\mathbb{R}^n_+$ into two parts: finite part (denote it by I) and infinite part (denote it by II). The existence of $\tilde{\mathbf{x}}$ is shown by verifying that $\mathbf{x}$ and $\mathbf{g}(\mathbf{0}^n|A)/n$ are in different parts.

Since $\mathbf{x} \in OP(A)$, we have $\mathbf{x} \in II \bigcup D$, i.e., $\mathbf{x}$ is in part II. Note that $\mathbf{z} \in OP(S)$ or $\mathbf{z} \notin S$, hence $S \subseteq I \cup D$. It is clearly that $L(\mathbf{g}(\mathbf{0}^n|A)/n) = 1$, thus $\mathbf{g}(\mathbf{0}^n|A)/n \in S$, which implies $\mathbf{g}(\mathbf{0}^n|A)/n \in I \cup D$, i.e., $\mathbf{g}(\mathbf{0}^n|A)/n$ is in part I. Since $\mathbf{g}(\mathbf{0}^n|A)/n \in I \cup D$, $\mathbf{x} \in II \cup D$ and ϕ is the segment determined by $\mathbf{x}$ and $\mathbf{g}(\mathbf{0}^n|A)/n$, we have $\phi \cap D \neq \emptyset$ which implies $\exists \tilde{\mathbf{x}} \in D \cap \phi$, i.e., $\tilde{\mathbf{x}} \in D$ such that $\tilde{\mathbf{x}}$ is a point on the segment ϕ.

Since $\mathbf{x} \in OP(\tilde{A})$, it is known that $\mathbf{x} \geq \mathbf{g}(\mathbf{0}^n|A)/n$, which implies $\mathbf{x} \geq \tilde{\mathbf{x}} \geq \mathbf{g}(\mathbf{0}^n|A)/n$. Hence $\tilde{\mathbf{x}} \in \tilde{\mathbf{D}}$,which implies

$$\mathbf{e}^T \cdot \mathbf{x} \geq \mathbf{e}^T \cdot \tilde{\mathbf{x}} \geq \min_{\mathbf{y} \in \tilde{D}} \mathbf{e}^T \cdot \mathbf{y}. \tag{23}$$

Because (23) holds for any given $\mathbf{x} \in OP(\tilde{A})$, applying (23) to (22) leads to

$$\mathbf{e}^T \cdot \mathbf{f}_{\mathbf{PM}}(A) \geq \min_{\mathbf{x} \in \tilde{D}} \mathbf{e}^T \cdot \mathbf{x}. \tag{24}$$

According to the definition of D, $\forall \mathbf{x} \in \tilde{D} \subseteq D$, there must be a $\mathbf{y} \in \tilde{S}$ such that $\mathbf{x}$ is on the segment with two end points $\mathbf{y}$ and $\mathbf{z}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{y}$ and $\mathbf{z}$: $\mathbf{x} = t(\mathbf{x})\mathbf{z} + (1 - t(\mathbf{x}))\mathbf{y}$, $t(\mathbf{x}) \in [0, 1]$. Since $\mathbf{y} \in \tilde{S}$, assume the kth coordinate of $\mathbf{y}$ is zero, i.e., $y_k = 0$. With $\mathbf{x} \in \tilde{D}$, we have $\mathbf{x} = t(\mathbf{x})\mathbf{z} + (1 - t(\mathbf{x}))\mathbf{y} \geq \mathbf{g}(\mathbf{0}^n|A)/n$, which implies,

$$t(\mathbf{x})z_k + 0 \geq g^k(\mathbf{0}^{n-1}|A)/n. \tag{25}$$

Since $\mathbf{z} \in OP(A)$, according to Assumption 2, we know that $z_k = g^k(\mathbf{z}_{-k}|A) \leq g^k(\mathbf{0}^{n-1}|A)$. Therefore, by (25), we have $t(\mathbf{x}) \geq 1/n$. Hence $\forall \mathbf{x} \in \tilde{D}$, we have

$$\mathbf{e}^T \cdot \mathbf{x} = \mathbf{e}^T \cdot (t(\mathbf{x})\mathbf{z} + (1 - t(\mathbf{x}))\mathbf{y}) = t(\mathbf{x})(\mathbf{e}^T \cdot \mathbf{z} - \mathbf{e}^T \cdot \mathbf{y}) + \mathbf{e}^T \cdot \mathbf{y}. \tag{26}$$

From $\mathbf{y} \in \tilde{S} \subseteq A$ and $\mathbf{e}^T \cdot \mathbf{z} = \max_{\mathbf{x} \in A} \mathbf{e}^T \cdot \mathbf{x}$, we have $\mathbf{e}^T \cdot \mathbf{z} \geq \mathbf{e}^T \cdot \mathbf{y}$. Hence, applying $t(\mathbf{x}) \geq 1/n$ to (26), it holds that

$$\mathbf{e}^T \cdot \mathbf{x} \geq \frac{1}{n}(\mathbf{e}^T \cdot \mathbf{z} - \mathbf{e}^T \cdot \mathbf{y}) + \mathbf{e}^T \cdot \mathbf{y}. \tag{27}$$

Since $\mathbf{y} \in \tilde{S}$, we have $1 = L(\mathbf{y}) = \sum_{i=1}^n (y_i / g^i(\mathbf{0}^{n-1}|A))$, which implies $1 \leq \sum_{i=1}^n (y_i / \min_{1 \leq j \leq n} g^j(\mathbf{0}^{n-1}|A))$, i.e.,

$$\min_{1 \leq j \leq n} g^j(\mathbf{0}^{n-1}|A) \leq \sum_{i=1}^n y_i = \mathbf{e}^\mathbf{T} \cdot \mathbf{y}. \tag{28}$$

According to $\mathbf{z} \in OP(A)$ together with Assumption 2, we have $\mathbf{z} = g(\mathbf{z}|A) = (g^1(\mathbf{z}_{-1}|A), \ldots, g^n(\mathbf{z}_{-n}|A)) \leq (g^1(\mathbf{0}^{n-1}|A), \ldots, g^n(\mathbf{0}^{n-1}|A)) = \mathbf{g}(\mathbf{0}^n|A)$. Hence,

$$\mathbf{e}^\mathbf{T} \cdot \mathbf{z} \leq \sum_{i=1}^n g^i(\mathbf{0}^{n-1}|A). \tag{29}$$

From (24), (27), (28) and (29), we have

$$\begin{aligned} E_A^e &= \frac{\mathbf{e}^T \cdot \mathbf{f_{PM}}(A)}{\mathbf{e}^T \cdot \mathbf{z}} \geq \min_{\mathbf{x}\in\tilde{D}} \frac{\mathbf{e}^T \cdot \mathbf{x}}{\mathbf{e}^T \cdot \mathbf{z}} \geq \min_{\mathbf{x}\in\tilde{D}} \left(\frac{1}{n} + \frac{n-1}{n}\frac{\mathbf{e}^T \cdot \mathbf{y}}{\mathbf{e}^T \cdot \mathbf{z}}\right) \\ &\geq \frac{1}{n} + \frac{n-1}{n} \cdot \frac{\min_{1\leq i\leq n} g^i(\mathbf{0}^{n-1}|A)}{\Sigma_{i=1}^n g^i(\mathbf{0}^{n-1}|A)}. \end{aligned} \tag{30}$$

Now for E_A^g, we know from definition that $E_A^e = E_A^g$ when $g^i(\mathbf{0^{n-1}}|A)$ are equal for all $i = 1, ..., n$. Together with E_A^g being scale invariant, we know that

$$E_A^g \geq \frac{1}{n} + \frac{n-1}{n} \cdot \frac{1}{n} = \frac{2n-1}{n^2}. \tag{31}$$

Finally for E_A^f, we estimate $\max_{\mathbf{x}\in A} \sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i$ directly. Since A is compact, there is an $\mathbf{x}^* \in A$ such that $\max_{\mathbf{x}\in A} \sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i = \sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i^*$. Note that $\mathbf{f_{PM}(A)}$ is Pareto optimal, there exists at least one coordinate m such that $f^m(A) \geq x_m^*$. And for any other $i \neq m$, we have from Theorem 2 that $f^i(A) \geq g^i(\mathbf{0^{n-1}}|A)/n \geq x_i^*/n$. Thus we have $\sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i^* \leq 1 + (n-1) \cdot n$. And hence we obtain

$$E_A^f = \frac{n}{\sum_{i=1}^n [f^i(A)]^{-1} \cdot x_i^*} \geq \frac{n}{n^2 - n + 1}. \tag{32}$$

This completes the proof.□

Remark. Based on Theorem 3, we obtain three types of POFs with respect to EPM solution:

$$POF_A^e(EPM) \doteq 1 - E_A^e \leq \frac{n-1}{n}\{1 - \frac{\min_{1\leq i\leq n} g^i(\mathbf{0}^{n-1}|A)}{\Sigma_{i=1}^n g^i(\mathbf{0}^{n-1}|A)}\}, \tag{33}$$

$$POF_A^g(EPM) \doteq 1 - E_A^g \leq \frac{(n-1)^2}{n^2} \tag{34}$$

and

$$POF_A^f(EPM) \doteq 1 - E_A^f \leq \frac{(n-1)^2}{n^2 - n + 1}. \tag{35}$$

According to Bertsimas et al. (2011) together with the facts that $E_A^f(N) \equiv 1$ and $E_A^f(KS) \equiv E_A^g(KS)$, we have (when $g^i(\mathbf{0^{n-1}}|A)$ are equal for all $i = 1, ..., n$)

$$POF_A^e(N) = POF_A^g(N) \leq \frac{(\sqrt{n}-1)^2}{n}, POF_A^f(N) \equiv 0 \tag{36}$$

and

$$POF_A^e(KS) = POF_A^g(KS) = POF_A^f(KS) \leq \frac{(n-1)^2}{(n+1)^2}. \tag{37}$$

Comparing the three solutions (Nash, KS and EPM), we know that the POFs of the Nash solution have the lowest upper bounds while those of the EPM solution have the highest upper bounds when the bargaining game is normalized. Therefore, the Nash solution is the most efficient one among the three solutions while the EPM solution is the most inefficient.

5 Examples

In this section, we give examples to show that the bounds in Theorem 3 are asymptotically tight. (In general, a bound is called asymptotically tight, if there exists a sequence of samples such that the bound is achieved by taking a limit of this sequence.) Before we present the examples, an extension of B^n_+ given by Calvo and Gutierrez (1994) is introduced.

In the original definition of the PM path, $g^i(\cdot|A)$ is in the class of $C^2(A^i)$, which excludes an interesting class of polygonal games (i.e., $OP(A)$ is the union of pieces of hyper-planes). For polygonal games, $g^i(\cdot|A)$ is C^2 almost everywhere in A^i. Calvo and Gutierrez (1994) extend B^n_+ to $B^n_+ \cup P_0$, where P_0 contains all the compact set A satisfying Assumptions 2 and $OP(A)$ being the union of pieces of hyper-planes. They have proved that there exists a unique EPM solution defined on $B^n_+ \cup P_0$. In this section, we focus on a subset of $B^n_+ \cup P_0$, i.e., $B^n_+ \cup P^*_0$, where P^*_0 contains all the bargaining set A that satisfies Assumptions 1, 2 and $OP(A)$ being the union of pieces of hyper-planes.

Now we briefly introduce the extension given by Calvo and Gutierrez (1994). Recall that a PM path is characterized by the ordinary differential equations (2), which depend on a vector-valued function $\mathbf{h}(x|A)$. For $A \in B^n_+$, $\mathbf{h}(x|A)$ can be obtained by the eigenvalue and eigenvector equation (1). However, for $A \in B^n_+ \cup P^*_0$, the eigenvalue and eigenvector equation (1) is not enough to obtain $\mathbf{h}(x|A)$ because $g^i(\cdot|A)$ may not be differentiable at some points in A^i. To overcome this difficulty, Calvo and Gutierrez (1994) change the part ii.2) of Definition 3 to that, the tangent vector of $C(A)$ at $\mathbf{x}$ is collinear to the tangent vector at $\mathbf{g}(\mathbf{x}|A)$ for $x \in C(A) \bigcap A$ almost everywhere.

Specifically, they denote $\triangle = \{\mathbf{h}|\mathbf{h} \geq \mathbf{0}^n, \mathbf{e}^T \cdot \mathbf{h} = 1\}$ and a function $\mathbf{d^x} : \triangle \to \mathbb{R}^n$ as follows,

$$\mathbf{d^x}(\mathbf{h}|A) = \lim_{t>0, t\to 0} \frac{\mathbf{g}(\mathbf{x} + t\mathbf{h}|A) - \mathbf{g}(\mathbf{x}|A)}{t}. \tag{38}$$

Calvo and Gutierrez (1994) show that $\mathbf{d^x}(\mathbf{h}|A) = (\mathbf{e}^T \cdot \mathbf{d^x}(\mathbf{h}|A))\mathbf{h}$ has a unique solution $\mathbf{h}^*(\mathbf{x}|A) = (h^*_1(\mathbf{x}|A), \ldots, h^*_n(\mathbf{x}|A)) \in \triangle$. Hence $\mathbf{d^x}(\mathbf{h}^*(\mathbf{x}|A)|A) = \lambda^*(\mathbf{x}|A)\mathbf{h}^*(\mathbf{x}|A)$, where $\lambda^*(\mathbf{x}|A) = \mathbf{e}^T \cdot \mathbf{d^x}(\mathbf{h}^*(\mathbf{x}|A)|A)$. Furthermore, they have proved that this approach is equivalent to solving the eigenvalue and eigenvector equation at every point $\mathbf{x} \in A$ such that $\mathbf{g}(\mathbf{x}|A)$ is differentiable at $\mathbf{x}$.

Similar to Theorems 1 and 2, we can prove the next two theorems (see Appendix).

Theorem 4 *$\forall A \in B^n_+ \cup P^*_0$, $\mathbf{x} = (x_1, \ldots, x_n)^T \in A$ and $\mathbf{x} \notin OP(A)$, we have $|\lambda^*(\mathbf{x}|A)| \leq n - 1$.*

Theorem 5 *$\forall A \in B^n_+ \cup P^*_0$, $\mathbf{f_{PM}}(A) \geq \mathbf{g}(\mathbf{0}^n|A)/n$.*

Applying Theorem 5 to the proof of Theorem 3, we have the following theorem.

Theorem 6 $\forall A \in B^n_+ \cup P^*_0$,

$$E^e_A \geq \frac{1}{n} + \frac{n-1}{n} \cdot \frac{\min_{1\leq i\leq n} g^i(\mathbf{0}^{n-1}|A)}{\Sigma^n_{i=1} g^i(\mathbf{0}^{n-1}|A)}, \tag{39}$$

$$E^g_A \geq \frac{2n-1}{n^2} \tag{40}$$

and

$$E^f_A \geq \frac{n}{n^2-n+1}. \tag{41}$$

Now, we give the examples to show the bounds in Theorem 6 are asymptotically tight. In text, we only give the results of our examples, all the calculations are in the Appendix.

Assume $0 < q^1 \leq q^2 \leq \ldots \leq q^n$, and for $s \in (0, 1)$ define the following linear functions

$$p_i(\mathbf{x}|s) = (1-s)\frac{x_i}{q^i} + s\Sigma^n_{j=1}\frac{x_j}{q^j}, \quad i = 1, \ldots, n. \tag{42}$$

For $0 < s < t < 1/n$, consider the following bargaining set

$$A(t, s) = \{\mathbf{x}|\mathbf{x} \geq 0, p_1(\mathbf{x}|t) \leq 1, p_i(\mathbf{x}|s) \leq 1, i = 2, \ldots, n\}. \tag{43}$$

Then we have $g^i(\mathbf{0}^{n-1}|A(t, s)) = q^i$ for all $0 < s < t < 1/n$ and $i = 1, ..., n$. The calculation in Appendix will show us that

$$\lim_{t\to 0}\lim_{s\to 0} E^e_{A(t,s)} = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{q^1}{\Sigma^n_{i=1} q^i}, \tag{44}$$

$$\lim_{t\to 0}\lim_{s\to 0} E^g_{A(t,s)} = \frac{2n-1}{n^2} \tag{45}$$

and

$$\lim_{t\to 0}\lim_{s\to 0} E^f_{A(t,s)} = \frac{n}{n^2-n+1}. \tag{46}$$

Therefore, the lower bounds in Theorem 6 are asymptotically tight. To be more intuitively, Figure 2 is given to show that these lower bounds are asymptotically tight when $n = 2$ $(A(t) = \lim_{s\to 0} A(t, s))$.

6 Conclusions

This paper aims to quantify the system efficiency of the EPM solution of Nash bargaining problem and to obtain asymptotically tight lower bounds for three measures. The lower bounds decreases as the number of players increases in a relationship of inverse proportion.

On the other hand, Pallaschke and Rosenmuller (2007) give another extension of the two-player PM solution for the cephoidal bargaining problem. They have shown that their approach satisfies the axioms suggested by Perles and Maschler (1981) in a sub-class of the cephoidal bargaining problems. Finding

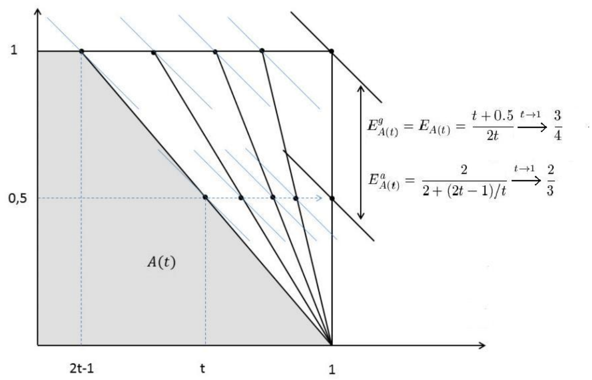

Fig. 2 Two dimensional examples to show the tightness.

lower bounds of this solution can be a future work.

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Appendix

Proof of Theorem 4

For a given $\mathbf{x}$ satisfying the condition in the theorem, it is clear that $g^i(\mathbf{x}_{-i}|A) > x_i$ because $\mathbf{x} \in A$ and $\mathbf{x} \notin OP(A)$. Consider the following matrix

$$\mathbf{J}_* = diag\{g^1(\mathbf{x}_{-1}|A) - x_1, \ldots, g^n(\mathbf{x}_{-n}|A) - x_n\}. \tag{47}$$

Then define function $\bar{\mathbf{g}}(t, \mathbf{k}) = (\bar{g}^1(t, \mathbf{k}), \ldots, \bar{g}^n(t, \mathbf{k}))^T = J_*^{-1}(\mathbf{g}(\mathbf{x} + tJ_*\mathbf{k}|A) - \mathbf{x})$ for $t \in [0, +\infty)$ and $\mathbf{k} \in \mathbb{R}^n_+$. (Note that $\bar{\mathbf{g}}(t, \mathbf{k})$ is well defined for $\mathbf{x} + tJ_*\mathbf{k} \in A$.)

Since $\mathbf{d}^{\mathbf{x}}(\mathbf{h}^*(\mathbf{x}|A)|A) = \lambda^*(\mathbf{x}|A) \cdot \mathbf{h}^*(\mathbf{x}|A)$, we have

$$\lambda^*(\mathbf{x}|A) \cdot \mathbf{h}^*(\mathbf{x}|A) = \lim_{t>0, t\to 0} \frac{\mathbf{g}(\mathbf{x} + t\mathbf{h}^*(\mathbf{x}|A)|A) - \mathbf{g}(\mathbf{x}|A)}{t}. \tag{48}$$

Hence, it holds that

$$\lambda^*(\mathbf{x}|A) \cdot J_*^{-1} \cdot \mathbf{h}^*(\mathbf{x}|A) = \lim_{t>0, t\to 0} \frac{J_*^{-1}\mathbf{g}(\mathbf{x} + tJ_*J_*^{-1}\mathbf{h}^*(\mathbf{x}|A)|A) - J_*^{-1}\mathbf{g}(\mathbf{x}|A)}{t},$$

which leads to

$$\lambda^*(\mathbf{x}|A) \cdot \mathbf{k}^* = \lim_{t>0, t\to 0} \frac{\bar{\mathbf{g}}(t, \mathbf{k}^*) - \bar{\mathbf{g}}(0, \mathbf{k}^*)}{t}, \tag{49}$$

where $\mathbf{k}^* = (k_1^*, \ldots, k_n^*)^T = J_*^{-1} \cdot \mathbf{h}^*(\mathbf{x}|A)$.

From Theorem 1, $g^i(\cdot|A)$ is concave in A^i, by which $\mathbf{e}^T \cdot \mathbf{g}(\mathbf{x}|A)$ is concave for all $\mathbf{x} \in A$. Therefore, we have, $\mathbf{e}^T \cdot \bar{\mathbf{g}}(t, \mathbf{k})$ is concave for all $\mathbf{k} \in \mathbb{R}^n_+$ such that $\mathbf{x} + tJ_*\mathbf{k} \in A$ (i.e., $\bar{\mathbf{g}}(t, \mathbf{k})$ is well defined), and $\mathbf{e}^T \cdot \bar{\mathbf{g}}(t, \mathbf{k})$ is also concave for all $t \in \mathbb{R}_+$ such that $\mathbf{x} + tJ_*\mathbf{k} \in A$.

Since $\mathbf{x} \notin OP(A)$, for sufficient small t, we have $\mathbf{x} + tJ_*\mathbf{k}^* \in A$ and $\mathbf{x} + t(\mathbf{e}^T J_*\mathbf{k}^*) \cdot \mathbf{e}^i \in A$, $\forall i = 1, 2, \ldots, n$. Hence, from (49), together with $\mathbf{e}^T \cdot \bar{\mathbf{g}}(t, \mathbf{k})$ being concave in $\mathbf{k}$ if it is well defined, we have

$$\begin{aligned}&\lambda^*(\mathbf{x}|A)\cdot\mathbf{e}^T\cdot\mathbf{k}^*\\&=\lim_{t>0,t\to 0}\frac{\mathbf{e}^T\cdot\bar{\mathbf{g}}(t,\mathbf{k}^*)-\mathbf{e}^T\cdot\bar{\mathbf{g}}(0,\mathbf{k}^*)}{t}\\&\geq\lim_{t>0,t\to 0}\frac{\sum_{i=1}^n\frac{k_i^*}{\mathbf{e}^T\cdot\mathbf{k}^*}(\mathbf{e}^T\cdot\bar{\mathbf{g}}(t,(\mathbf{e}^T\cdot\mathbf{k}^*)\mathbf{e}^i)-\mathbf{e}^T\cdot\bar{\mathbf{g}}(0,\mathbf{k}^*))}{t}.\end{aligned}\tag{50}$$

Since the last term in (50) is well defined for all $t\in(0,(\mathbf{e}^T\cdot\mathbf{k}^*)^{-1}]$, it is known that $\bar{\mathbf{g}}(t,(\mathbf{e}^T\cdot\mathbf{k}^*)\mathbf{e}^i)$ is concave for t. Hence, we have from (50)

$$\begin{aligned}&\lambda^*(\mathbf{x}|A)\cdot\mathbf{e}^T\cdot\mathbf{k}^*\\&\geq\lim_{t>0,t\to 0}\frac{\sum_{i=1}^n\frac{k_i^*}{\mathbf{e}^T\cdot\mathbf{k}^*}(\mathbf{e}^T\cdot\bar{\mathbf{g}}(t,(\mathbf{e}^T\cdot\mathbf{k}^*)\mathbf{e}^i)-\mathbf{e}^T\cdot\bar{\mathbf{g}}(0,\mathbf{k}^*))}{t}\\&\geq\frac{\sum_{i=1}^n\frac{k_i^*}{\mathbf{e}^T\cdot\mathbf{k}^*}(\mathbf{e}^T\cdot\bar{\mathbf{g}}((\mathbf{e}^T\cdot\mathbf{k}^*)^{-1},(\mathbf{e}^T\cdot\mathbf{k}^*)\mathbf{e}^i)-\mathbf{e}^T\cdot\bar{\mathbf{g}}(0,\mathbf{k}^*))}{(\mathbf{e}^T\cdot\mathbf{k}^*)^{-1}}\\&=\sum_{i=1}^n k_i^*(\mathbf{e}^T\cdot\bar{\mathbf{g}}(1,\mathbf{e}^i)-\mathbf{e}^T\cdot\bar{\mathbf{g}}(0,\mathbf{k}^*))\\&=\sum_{i=1}^n k_i^*(\mathbf{e}^T\cdot J_*^{-1}(\mathbf{g}(\mathbf{x}+J_*\mathbf{e}^i|A)-\mathbf{x})-\mathbf{e}^T\cdot J_*^{-1}(\mathbf{g}(\mathbf{x}|A)-\mathbf{x}))\\&=\sum_{i=1}^n k_i^*(1-n).\end{aligned}\tag{51}$$

The last equality holds because $\mathbf{x}+J_*\mathbf{e}^i\in OP(A)$ (hence $\mathbf{g}(\mathbf{x}+J_*\mathbf{e}^i|A)=\mathbf{x}+J_*\mathbf{e}^i$) and $J_*^{-1}(\mathbf{g}(\mathbf{x}|A)-\mathbf{x})=\mathbf{e}^T$. Therefore, we have obtained

$$\lambda^*(\mathbf{x}|A)\geq-(n-1).\tag{52}$$

Since $\mathbf{g}(\mathbf{x}|A)$ is decreasing, from (49), it is easy to see $\lambda^*(\mathbf{x}|A)\leq 0$. Hence, we have proved that $|\lambda^*(\mathbf{x}|A)|\leq n-1$.□

Proof of Theorem 5

Proof : Here, we consider the same function $\mathbf{W}(u|A)$ in Theorem 2. Recall that the EPM solution of A in Theorem 2 is $\mathbf{f}_{PM}(A)=\eta(u_{PM}|A)$. Then, we have already known that it is sufficient to show $\mathbf{W}(u|A)$ is non-decreasing. However, we are not able to prove $\mathbf{W}'(u|A)\geq\mathbf{0}^n$ because $\mathbf{W}(u|A)$ may not be differentiable. Here, we adopt another approach, that is to prove the right-sided derivative of $\mathbf{W}(u|A)$ is non-negative, i.e., to prove

$$\mathbf{W}'_+(u|A)\doteq\lim_{t>0,t\to 0}\frac{\mathbf{W}(u+t|A)-\mathbf{W}(u|A)}{t}\geq\mathbf{0}^n,\tag{53}$$

for all $u\in[0,u_{PM})$. (In a lemma after this theorem, we show that this condition guarantees $\mathbf{W}(u|A)$ being non-decreasing.)

From Theorem 2, we have

$$\mathbf{W}'_+(u|A)=(n-1)\eta'_+(u|A)+\lim_{t>0,t\to 0}\frac{\mathbf{g}(\eta(u+t)|A)-\mathbf{g}(\eta(u)|A)}{t}.\tag{54}$$

Note that $\eta(u|A)$ is strictly increasing, hence $\eta'_+(u|A)=\mathbf{h}^*(\eta(u|A)|A)\geq\mathbf{0}^n$. From $\eta'_+(u|A)\geq\mathbf{0}^n$ and $\eta'_+(u|A)\neq\mathbf{0}^n$, it follows that

$$\begin{aligned}&\lim_{t>0,t\to 0}\frac{\mathbf{g}(\eta(u+t)|A)-\mathbf{g}(\eta(u)|A)}{t}\\&\quad=\mathbf{d}^{\eta(u|A)}(\eta'_+(u|A)|A)=\mathbf{d}^{\eta(u|A)}(\mathbf{h}^*(\eta(u|A)|A)|A)\\&\quad=\lambda^*(\eta(u|A)|A)\mathbf{h}^*(\eta(u|A)|A).\end{aligned}\tag{55}$$

Substituting (55) into (54) leads to

$$\mathbf{W}'_+(u|A) = (n-1+\lambda^*(\eta(u|A)|A))\mathbf{h}^*(\eta(u|A)|A). \tag{56}$$

Then, from Theorem 4, it is known that $\mathbf{W}'_+(u|A) \geq \mathbf{0}^n$, which completes the proof.□

Lemma 1 *If $f : [a, b) \to \mathbb{R}$ is a continuous function, and $f'_+(x) \geq 0$ for all $x \in [a, b)$, then f is increasing in $[a, b)$.*

Proof : Consider $\overline{f}(x) = f(x) + \varepsilon x$ for $x \in [a, b)$ and $\varepsilon > 0$. Then, $\overline{f}'_+(x) \geq \varepsilon$. We first show that $\overline{f}$ is increasing in $[a, b)$.

If $\overline{f}$ is not increasing in $[a, b)$, then there are $c \in [a, b)$ and $d \in [a, b)$ such that $c < d$ and $\overline{f}(c) > \overline{f}(d)$. The continuity of $\overline{f}(x)$ in $[c, d]$ implies that there exists $\theta \in [c, d]$ such that $\overline{f}(\theta) = \sup_{x\in[c,d]} \overline{f}(x)$. Since $\overline{f}(c) > \overline{f}(d)$, we have $\theta \neq d$, hence $\varepsilon \leq \overline{f}'_+(\theta) = \lim_{t>0,t\to 0} \frac{\overline{f}(\theta+t)-\overline{f}(\theta)}{t} \leq 0$, which is in contradiction with $\varepsilon > 0$! This means that $\overline{f}$ is increasing in $[a, b)$.

From the increasing property of $\overline{f}$ in $[a, b)$, $\forall a \leq c \leq d < b$, we have $\overline{f}(d) \geq \overline{f}(c)$, i.e., $f(d) \geq f(c)+\varepsilon(c-d)$. Let $\varepsilon \to 0$, it follows that $f(d) \geq f(c)$. This means that f is increasing in $[a, b)$.□

Calculation of the examples

We need to show that $A(t, s) \in P_0^*$, i.e., $A(t, s)$ satisfies Assumptions 1, 2 and $OP(A(t, s))$ being the union of pieces of hyper-planes. It is easy to see from the definition of $A(t, s)$ that $A(t, s)$ is convex, compact, comprehensive and $OP(A(t, s))$ is the union of pieces of hyper-planes. Hence, we only need to verify that $A(t, s)$ satisfies Assumption 2. According to the definition of $A(t, s)$, $\forall \mathbf{x} \in A(t, s)$, we have

$$g^1(\mathbf{x}_{-1}|A(t,s)) = \min\left\{q^1 - t\sum_{i\neq 1}\frac{q^1 x_i}{q^i}, \min_{j\neq 1}\left\{\frac{q^1 - q^1 x_j}{s} - \sum_{i\neq 1,j}\frac{q^1 x_i}{q^i}\right\}\right\}.$$

And $\forall k = 2, \ldots, n$,

$$g^k(\mathbf{x}_{-k}|A(t,s)) =$$
$$\min\left\{q^k - s\sum_{i\neq k}\frac{q^k x_i}{q^i}, \frac{q^k - q^k x_1}{t} - \sum_{i\neq k,1}\frac{q^k x_i}{q^i}, \min_{j\neq 1,k}\left\{\frac{q^k - q^k x_j}{s} - \sum_{i\neq k,j}\frac{q^k x_i}{q^i}\right\}\right\}.$$

It is easy to see that $g^k(\mathbf{0}^{n-1}|A(t,s)) = q^k$ and $g^k(\cdot|A(t,s))$ is continuous and strictly decreasing, $\forall k = 1, \ldots, n$. For any given $k \in \{1, \ldots, n\}$, we show that $\mathbf{x} \in OP(A(t,s)) \Longleftrightarrow x_k = g^k(\mathbf{x}_{-k}|A(t,s))$.

If $\mathbf{x} \in OP(A(t,s))$, then $p_1(\mathbf{x}|t) \leq 1, p_i(\mathbf{x}|s) \leq 1, i = 2, \ldots, n$, and at least one of them is equality, which implies $x_k = g^k(\mathbf{x}_{-k}|A(t,s))$.

If $x_k = g^k(\mathbf{x}_{-k}|A(t,s))$, then $p_1(\mathbf{x}|t) \leq 1, p_i(\mathbf{x}|s) \leq 1, i = 2, \ldots, n$, and at least one of them is with equality. Assume $p_l(\mathbf{x}|r) = 1$ where $1 \leq l \leq n$ and $r \in \{s, t\}$. If $\mathbf{x} \notin OP(A(t,s))$, then $\exists \mathbf{y} \in A(t,s)$ such that $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$. Since each coefficient in $p_l(\cdot|r)$ is strictly positive, we have $p_l(\mathbf{y}|r) > 1$, which leads to a contradiction with $\mathbf{y} \in A(t,s)$. Hence, $\mathbf{x} \in OP(A(t,s))$.

Therefore, $A(t, s)$ satisfies Assumption 2, which implies $A(t, s) \in P_0^*$. Since $g^k(\mathbf{0}^{n-1}|A(t,s)) = q^k$ for $k = 1, \ldots, n$, it is known that the maximum achievable profit for the kth player is q^k.

A two dimensional example of $A(t, s)$ is shown in Figure 3. Since $OP(A(t, s))$ is the union of pieces of hyper-planes, we know that $g^i(\cdot|A(t,s))$ is piecewise linear in $A^i(s,t)$. Therefore, $\mathbf{G}(\mathbf{x}|A(t,s)) = \partial\mathbf{g}(\mathbf{x}|A(t,s))/\partial\mathbf{x}$ is a piecewise constant matrix in $A(t, s)$ (i.e., $\mathbf{G}(\mathbf{x}|A(t,s))$ is a constant matrix in each piece of $A(t, s)$). Hence, the tangent vector $\mathbf{h}^*(\mathbf{x}|A(t,s))$ of $C(A(t,s))$ is a piecewise constant vector, which indicates that $C(A(t,s))$ is piecewise linear. Because $C(A(t,s))$ is piecewise linear, finding the points on the PM path of $A(t, s)$ which are not differentiable is enough to determine the PM path $C(A(t,s))$.

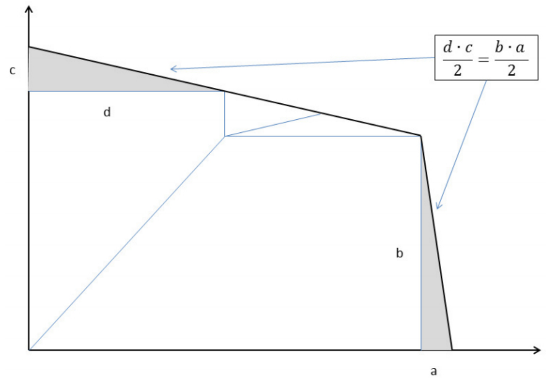

Fig. 3 A two dimensional example of $A(t,s)$.

Since $C(A(t,s))$ starts from $\mathbf{0}^n$, we need to calculate $\mathbf{G}(\mathbf{0}^n|A(t,s))$ and $\mathbf{h}(\mathbf{0}^n|A(t,s))$. Note that in a small local area of $\mathbf{0}^n$, it follows that

$$\begin{aligned} g^1(\mathbf{x}_{-1}|A(t,s)) &= 1 - t\sum_{j\neq 1}\frac{q^1 x_j}{q^j}, \\ g^i(\mathbf{x}_{-i}|A(t,s)) &= 1 - s\sum_{j\neq i}\frac{q^i x_j}{q^j}, 2 \leq i \leq n. \end{aligned} \tag{57}$$

Hence

$$\begin{aligned} \mathbf{G}(\mathbf{0}^n|A(t,s)) &= \begin{bmatrix} 0 & \frac{-tq^1}{q^2} & \dots & \frac{-tq^1}{q^n} \\ \frac{-sq^2}{q^1} & 0 & \dots & \frac{-sq^2}{q^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-sq^n}{q^1} & \frac{-sq^n}{q^2} & \dots & 0 \end{bmatrix}, \\ \mathbf{h}(\mathbf{0}^n|A(t,s)) &= \left(1, \frac{q^2\epsilon}{q^1}, \dots, \frac{q^n\epsilon}{q^1}\right)^T, \end{aligned} \tag{58}$$

where

$$\epsilon = \frac{(n-2)s + \sqrt{(n-2)^2s^2 + 4(n-1)st}}{2(n-1)t}. \tag{59}$$

Then, the PM path $C(A(t,s))$ starts from $\mathbf{0}^n$ and moves along the direction $\mathbf{h}(\mathbf{0}^n|A(t,s))$ until reaching the point $\mathbf{y}$ where $g^i(\cdot|A(t,s))$ is not differentiable at $\mathbf{y}_{-i}$ for $i \in \{2,\dots,n\}$. Then, $\mathbf{y}$ satisfies that $\forall i \in \{2,\dots,n\}$, $(\mathbf{y}_{-i} : g^i(\mathbf{y}_{-\mathbf{i}}|A(t,s)))$ is in the intersection of two hyper-planes $p_1(\mathbf{x}|t) = 1$ and $p_i(\mathbf{x}|s) = 1$. Consequently, we have

$$\mathbf{y} = bq^1\mathbf{h}(\mathbf{0}^n|A(t,s)) = (bq^1, b\epsilon q^2, \dots, b\epsilon q^n)^T, \tag{60}$$

where

$$b = \frac{1-t}{1 - st + t(1-s)(n-2)\epsilon}. \tag{61}$$

For any $\mathbf{x} \in A(t,s)$ and $\mathbf{x} > \mathbf{y}$, we have

$$\begin{aligned} g^1(\mathbf{x}_{-1}|A(t,s)) &= q^1 - t\sum_{j\neq 1}\frac{q^1 x_j}{q^j}, \\ g^i(\mathbf{x}_{-i}|A(t,s)) &= \frac{q^i}{t} - \sum_{j\neq 1,i}\frac{q^i x_j}{q^j} - \frac{q^i x_1}{tq^1}, 2 \leq i \leq n. \end{aligned} \tag{62}$$

The above leads to

$$
\begin{aligned}
\mathbf{G}(\mathbf{x}|A(t,s)) &= \begin{bmatrix} 0 & \frac{-tq^1}{q^2} & \cdots & \frac{-tq^1}{q^n} \\ \frac{-q^2}{tq^1} & 0 & \cdots & \frac{-q^2}{q^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-q^n}{tq^1} & \frac{-q^n}{q^2} & \cdots & 0 \end{bmatrix}, \\
\mathbf{h}(\mathbf{x}|A(t,s)) &= \left(1, \frac{q^2}{q^1 t}, \ldots, \frac{q^n}{q^1 t}\right)^T.
\end{aligned} \tag{63}
$$

Then, the PM path starts again from $\mathbf{y}$ and moves along the direction $\mathbf{h}(\mathbf{x}|A(t,s))$ until reaching the EPM solution $\mathbf{f_{PM}}(A(t,s))$ in the hyper-plane $p_1(\mathbf{x}|t) = 1$. Hence,

$$\mathbf{f_{PM}}(A(t,s)) = \mathbf{y} + cq^1\mathbf{h}(\mathbf{x}|A(t,s)), \tag{64}$$

where

$$c = \frac{1 - b - t\epsilon b(n-1)}{n}. \tag{65}$$

So far, we have found the EPM solution $\mathbf{f_{PM}}(A(t,s))$. Next, we estimate the system efficiency.

Assume that $\mathbf{z}$ is the intersection of the hyper-planes $p_1(\mathbf{x}|t) = 1$ and $p_i(\mathbf{x}|s) = 1$ for all $i = 2, \ldots, n$. Then, we have

$$\mathbf{z} = \frac{\left((1 + (n-2)s - (n-1)t)q^1, (1-s)q^2, \ldots, (1-s)q^n\right)^T}{1 + (n-2)s - (n-1)st}. \tag{66}$$

Let $s \to 0$, we have $\epsilon \to 0$, $b \to 1-t$ and $c \to t/n$, then let $t \to 0$ we have $\mathbf{f_{PM}}(A(t,s)) \to (q^1, q^2/n, ..., q^n/n)$ and $\mathbf{z} \to (q^1, q^2, ..., q^n)$.

Therefore, we have the following inequalities

$$\lim_{t\to 0}\lim_{s\to 0} E^e_{A(t,s)} \le \lim_{t\to 0}\lim_{s\to 0} \frac{\mathbf{e}^T \cdot \mathbf{f_{PM}}(A(t,s))}{\mathbf{e}^T \cdot \mathbf{z}} = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{q^1}{\sum_{i=1}^n q^i}, \tag{67}$$

$$\lim_{t\to 0}\lim_{s\to 0} E^g_{A(t,s)} \le \lim_{t\to 0}\lim_{s\to 0} \frac{\sum_{i=1}^n (q^i)^{-1} \cdot f^i(A)}{\sum_{i=1}^n (q^i)^{-1} \cdot z_i} = \frac{2n-1}{n^2} \tag{68}$$

and

$$\lim_{t\to 0}\lim_{s\to 0} E^f_{A(t,s)} \le \lim_{t\to 0}\lim_{s\to 0} \frac{n}{\sum_{i=1}^n [f^i(A)]^{-1} \cdot z_i} = \frac{n}{n^2 - n + 1}. \tag{69}$$

Together with Theorem 6, we know that the above inequalities become equalities, hence the bounds given in Theorem 6 are asymptotically tight.

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