

Revisiting Prospect Theory and the Newsvendor Problem

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Abstract

Many experimental studies have demonstrated that human decision-makers exhibit the pull-to-center effect in newsvendor decision. It has been shown in the literature that prospect theory with a decision-dependent reference point can predict the pull-to-center effect for the newsvendor problem by assuming a uniform distribution of demand. In this paper, we prove this result for a general case: prospect theory with a decision-independent reference point can predict the pull-to-center effect for the newsvendor problem with a general distribution of demand.

Keywords: Prospect theory; Newsvendor; Pull-to-center effect; Reference point.

1. Introduction

In the past decade, behavioral operations management has garnered an increasing amount of research interest. In a pioneering work, Schweitzer and Cachon [1] conducted experiments to investigate the behavior of human decision-makers based on newsvendor settings. They observed that the order quantity of subjects exhibited a “pull-to-center” effect, i.e., the order quantity was likely to fall in the range between the 0.5 fractile of the demand distribution and the optimal solution. According to the newsvendor model, settings with a critical fractile in the range $[0, 0.5)$ are classified as low-profit

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margins, whereas those with a critical fractile in the range $(0.5, 1]$ are classified as high-profit margins. The pull-to-center effect then represents the case where the order quantity is too high for a low-profit-margin setting and too low for a high-profit-margin setting. Using experimental data, Schweitzer and Cachon [1] documented that prospect theory cannot predict the behavior of subjects. Following Schweitzer and Cachon [1], many researchers conducted numerous experiments to observe the pull-to-center effect.

Recently, Nagarajan and Shechter [2] use a model of prospect theory with a power-type utility function to discuss its effectiveness in predicting the pull-to-center effect. For a low-profit-margin setting with only positive profit, they theoretically prove that the order quantity is lower than the optimal solution. For a high-profit-margin setting with only positive profit, they numerically show that the order quantity is higher than the optimal value. They then claim that prospect theory cannot explain the pull-to-center effect.

In both [1] and [2], they assume the reference point to be zero, i.e., take the status quo as reference. However, many evidences imply different possibilities, e.g., as suggested by Heath et al. [3], that goals serve as reference points is quite appropriate and can explain lots of empirical results. Taking into consideration a non-zero reference point, Zhao and Geng [4] show that prospect theory can satisfactorily predict the pull-to-center effect through numerical calculations (but no analytical results). Furthermore, Long and Nasiry [5] use a model with a decision-dependent reference point that is a specified function of order quantity. For uniform distribution of demand, they theoretically prove that the model can predict the pull-to-center effect.

In this discussion, we extend the analytical results in the literature to a general case from two aspects: general distribution of demand, and decision-independent reference point. Relaxing the demand from a uniform distribution to a general distribution is apparently significant. On the other hand, adopting a decision-independent reference point is a popular scenario in decision making (see, e.g., the literature on experimental economics, [6, 7, 8]). When a person makes a decision, his/her reference point may be related to contextual parameters, but should be less related to the decision he/she made (see, e.g., [7]). From this perspective, we need to investigate the decision-independent reference point, although a decision-dependent reference point may have the advantage of being able to facilitate model analysis by specially setting a function type. Consequently, the general case in the sense of a general distribution of demand and a decision-independent reference point has the value for developing further studies on the topic. (In a working paper,

Uppari and Hasija [9] also considered a general distribution of demand, but with a reference point that is associated with the mean demand. Intuitively, a newsvendor may be contextual in terms of low- or high-profit-margin setting to set his/her reference point accordingly, rather than anchoring on the mean demand only. Hence, in contrast, in our analysis, we reveal a reference point (possibly a set of reference points) that can lead to the pull-to-center effect.)

2. Model Analysis

The basic setting is the classical newsvendor model. Suppose that the marginal cost is $w(> 0)$ and the selling price is $p(> w)$. The demand D is a random variable with support $[\underline{d}, \bar{d}]$ where $\bar{d} > \underline{d} \geq 0$, and follows distribution function $F(\cdot)$ with density function $f(\cdot)$. (All the results in this paper also hold for the demand distribution with support $[\underline{d}, +\infty)$.) Assume that F and f are differentiable and $F(\underline{d}) = 0$; $\bar{F}(x) = 1 - F(x)$. Moreover, let $F_{0.5} = F^{-1}(0.5)$ denote the demand corresponding to the 0.5 fractile of F . Apparently, for any distribution with the symmetric property at its mean μ , such as a uniform distribution, a normal distribution, etc., $F_{0.5} = \mu$. In this paper, we do not require that $F_{0.5} = \mu$, but general cases can apply.

The newsvendor makes a decision of order quantity q to maximize the following expected profit function:

$$\Pi(q) = pE \min(D, q) - wq.$$

It is easy through standard analyses to obtain the optimal solution $q_c = F^{-1}(1 - w/p)$. This standard optimal solution can be referred to the benchmark for decision makers to be perfectly rational.

However, it has been recognized that human decision-makers are not perfectly rational when making decisions. Kahneman and Tversky [10] developed the so-called prospect theory to characterize the behavior of human decision-makers. They proposed a value function that is: 1) defined by deviations from the reference point, 2) generally concave for gains and commonly convex for losses, and 3) steeper for losses than for gains. Thus, in general, the value function presents an S-shape.

For the newsvendor setting, if the realization of stochastic demand D is x , the resultant profit for a decision q is given by

$$\pi(q, x) = p \min(x, q) - wq.$$

Suppose that the newsvendor has reference point r . We assume that the reference point is independent of decision q . Also note that the reference point can be either negative or positive. To express the value function in prospect theory, we use the exponential-type utility function $u(y) = 1 - e^{-\alpha y}$, where coefficient $\alpha(> 0)$ characterizes the risk attitude. (It is well-known that the power function and the exponential function are most popular for calculating utility. In this study, we found that the exponential-type utility function can relatively facilitate the model analysis, whereas the power-type utility function seems to be intractable for model analyses.) The value function can then be expressed as the following S-shaped utility function:

$$U(\pi|r) = \begin{cases} u(\pi - r), & \text{if } \pi > r, \\ -\lambda u(r - \pi), & \text{if } \pi \leq r, \end{cases}$$

where the coefficient $\lambda(\geq 1)$ characterizes the degree of loss aversion.

The objective of the newsvendor is to determine order quantity $q \in [\underline{d}, \bar{d}]$ to maximize the expected utility, i.e.,

$$\max_{q \in [\underline{d}, \bar{d}]} V(q|r) = EU(\pi(q, D)|r). \quad (1)$$

We denote the set of optimal solutions of problem (1) by $Q^*(r)$. In the subsequent analyses, we show that prospect theory can predict the pull-to-center effect. For this purpose, we need to prove that there exists a value r (possibly a set of r), using which the optimal solution q^* (in $Q^*(r)$) to problem (1) lies between $F_{0.5}$ and q_c , i.e., between the 0.5 fractile of the demand distribution and the standard optimal order quantity.

Let $I_0 = [p\underline{d} - w\underline{d}, p\bar{d} - w\bar{d}]$, which is the set of all possible profits for the newsvendor. For the case of the reference point $r \notin I_0$, problem (1) degenerates to a risk-averse (or risk-seeking) newsvendor problem due to always gain (or always loss) over $q \in [\underline{d}, \bar{d}]$, which has been widely discussed in the literature. Therefore, we only focus on the case that $r \in I_0$ in this paper.

Before the analyses, we introduce the following notation:

$$I_1 = [p\underline{d} - w\underline{d}, p\bar{d} - w\bar{d}], \quad I_2 = [p\underline{d} - w\bar{d}, p\bar{d} - w\underline{d}],$$

$$q_1(r) = \frac{r}{p - w}, \quad q_2(r) = \frac{p\underline{d} - r}{w}, \quad q_0(r) = \max(q_1(r), q_2(r)),$$

$$B = \left\{ r \mid r \in I_2 \text{ and } \left. \frac{dV(q|r)}{dq} \right|_{q=q_2(r)} < 0 \right\}.$$

Here, $q_1(r)$ is the breakeven point for reference point $r \in I_1$, i.e., if the order quantity is $q_1(r)$, the newsvendor's highest possible profit equals the reference point r . For reference point $r \in I_2$, $q_2(r)$ is the breakeven point, i.e., if the order quantity is $q_2(r)$, the newsvendor's lowest possible profit equals r (when the realized demand is \underline{d}). B is the set formed by all r in I_2 such that $V(q|r)$ is decreasing near the point $q = q_2(r)$. Other than the above notation, for a given $r \in I_0$, we further define $Q_0(r)$ to be the solution set of $\max_{q \in [q_0(r), \bar{d}]} V(q|r)$. (Possibly, $Q_0(r)$ may be an empty set for some r .)

With the above preliminaries, we present our main results. All proofs are presented in Appendix A (see the Online Supplements).

Proposition 1. *For any given $r \in I_0$, $V(q|r)$ is decreasing or unimodal when $q \in [q_0(r), \bar{d}]$.*

This proposition guarantees that the solution set of $\max_{q \in [q_0(r), \bar{d}]} V(q|r)$ is an interval or a singleton set. With Proposition 1, we have the following result.

Proposition 2. (1) *If $r \in [(p-w)\underline{d}, (p-w)q_c] \cup (I_2 \setminus B)$, then $Q^*(r) \subseteq Q_0(r) \cup \{q_0(r)\}$;*
(2) *If $r \in ((p-w)q_c, (p-w)\bar{d}]$, then $Q^*(r) \subseteq Q_0(r) \cup (q_c, q_0(r)]$;*
(3) *If $r \in B$, then $Q^*(r) = \{q^*\}$ is a singleton set, and q^* satisfies $M(q^*) = 0$, where*

$$M(q) = -w \int_0^q u'(px - wq - r)f(x)dx + (p-w)u'(pq - wq - r)\bar{F}(q).$$

The above proposition provides us with a preliminary view of the solution to problem (1). The first part indicates that the optimal order quantity is greater than or equal to $q_0(r)$, conditioned on r . The second part indicates that the optimal order quantity is greater than q_c , conditioned on r . The third part implies the case where, when the reference point is under a certain condition, the problem degenerates to a risk-averse problem and, hence, the optimal order quantity is smaller than q_c .

Theorem 1. *For a high-profit-margin setting, i.e., $p > 2w$, if $1 \leq \lambda \leq \frac{p-w}{w}$, there exists a nonempty set $A_H \subseteq [(p-w)\underline{d}, (p-w)F_{0.5})$, such that for $r \in A_H$, $Q^*(r) = \{q^*\}$ is a singleton set with $q^* \in (F_{0.5}, q_c)$.*

It has been shown that human decision-makers exhibit a bias toward loss aversion when making decisions [11, 12]. The loss-averse attitude is characterized by the coefficient $\lambda \geq 1$. Then, Theorem 1 indicates that for a reasonable reference point, the order quantity is greater than the 0.5 fractile of the demand distribution and lower than the standard optimal order quantity, which is consistent with the experimental observations in the literature. Thus, prospect theory predicts the pull-to-center effect for high-profit-margin settings.

Theorem 2. *For a low-profit-margin setting, i.e., $p < 2w$, if $f'(x) \geq 0$ for all $x \in [\underline{d}, F_{0.5}]$, there exists a nonempty set $A_L \subseteq (p\underline{d} - wq_c, (p - w)F_{0.5})$, such that $Q^*(r) \subseteq (q_c, F_{0.5})$ for $r \in A_L$.*

Many common distributions, such as a uniform distribution, a normal distribution, a logistic distribution, etc., satisfy condition $f'(x) \geq 0$ for all $x \in [\underline{d}, F_{0.5}]$. With this condition, Theorem 2 indicates that for a reasonable reference point, the order quantity is greater than q_c and less than $F_{0.5}$, which is consistent with the experimental observations in the literature. Thus, prospect theory can predict the pull-to-center effect for low-profit-margin settings.

The following algorithm provides the detailed calculation steps for A_H and A_L , where

$$H(q, r) = -\lambda w \int_{\underline{d}}^{\frac{r+wq}{p}} u'(r+wq-px)f(x)dx - w \int_{\frac{r+wq}{p}}^q u'(px-wq-r)f(x)dx + (p-w)u'(pq-wq-r)\bar{F}(q).$$

Furthermore, we define $\inf \emptyset = +\infty$ in this paper.

Algorithm 1.

Step 1: If $p > 2w$, go to Step 2; Otherwise, go to Step 4.

Step 2: If $H(F_{0.5}, (p - w)\underline{d}) > 0$, let $r_1 = \inf\{r | r \in ((p - w)\underline{d}, (p - w)F_{0.5}) \text{ and } H(q_c, r) = 0\}$, $r_2 = \inf\{r | r \in ((p - w)\underline{d}, (p - w)F_{0.5}) \text{ and } H(F_{0.5}, r) = 0\}$, and set $A_H = [(p - w)\underline{d}, \min(r_1, r_2, (p - w)F_{0.5}))$, stop; Otherwise, go to Step 3.

Step 3: Let $r_3 = \sup\{r | r \in [(p - w)\underline{d}, (p - w)F_{0.5}] \text{ and } H(F_{0.5}, r) = 0\}$, $r_4 = \inf\{r | r \in [r_3, (p - w)F_{0.5}] \text{ and } H(q_c, r) = 0\}$, and set $A_H = (r_3, \min((p - w)F_{0.5}, r_4))$, stop.

Step 4: If $H(q_c, (p - w)q_c) > 0$, let $r_5 = \sup\{r | r \in (p\underline{d} - wq_c, (p - w)q_c) \text{ and } H(q_c, r) = 0\}$, $r_6 = \inf\{r | r \in (r_5, (p - w)F_{0.5}] \text{ and } H(F_{0.5}, r) = 0\}$, and set $A_L = (r_5, \min(r_6, (p - w)F_{0.5}))$, stop; Otherwise, go to Step 5.

Step 5: Let $r_7 = \inf\{r | r \in ((p - w)q_c, (p - w)F_{0.5}] \text{ and } H(F_{0.5}, r) = 0\}$, and set $A_L = [(p - w)q_c, \min(r_7, (p - w)F_{0.5}))$, stop.

The validity of the above calculation steps can be referred to the proofs of Theorem 1 and Theorem 2.

Then, for any given reference point in A_H (or in A_L), we can search the corresponding optimal order quantity with consideration of Proposition 1 and Proposition 2.

We end this section with two observations. First, prospect theory consists of three behavioral parameters: risk attitude α , loss attitude λ , and reference point r . We have proved the existence of reference point r within a certain range that leads to the optimal solution q^* of problem (1) lying between $F_{0.5}$ and q_c . Note that this existence property depends on system parameters (w , p , and D), and is true over reasonable scopes of behavioral parameters $\alpha(> 0)$ and $\lambda(\geq 1)$. In an experiment, subjects may have their individual behavioral parameters α and λ . Hence, their reference points can be different, which can either fall into the ranges A_H and A_L or be out of these ranges. Second, in the literature, almost all relevant studies have been conducted using demand distributions that are symmetric at the mean, i.e., $F_{0.5} = \mu$. The case $F_{0.5} = \mu$ is a special case of our model. Consequently, our results can predict the pull-to-center effect not only for such a special case, but also for more general cases.

3. Examples

In the experimental study in [1], the authors used parameters $p = 12$, $D \sim U[0, 300]$, and $w = 9$ for a low-profit-margin setting and $w = 3$ for a high-profit-margin setting. (Thus, the critical fractiles were 0.25 and 0.75 and the standard optimal solutions were 75 and 225 for the two settings, respectively.) For the low-profit-margin setting, the feasible lowest profit was -2700 whereas the feasible highest profit was 900. For the high-profit-margin setting, the feasible lowest profit was -900 whereas the feasible highest profit was 2700. From their experimental data, they observed that the subjects' decisions exhibited the pull-to-center effect. However, they were not sure whether prospect theory could predict this effect. Following their study, many researchers conducted similar experiments prompted by varying motivations.

Using our model, we carry out calculations for this example. According to [11], we set the coefficient of loss aversion to the mostly like value, i.e., $\lambda = 2$.

Table 1: Scope of reference point that leads to the pull-to-center effect ($D \sim U[0, 300]$, $\lambda = 2$).

α	High-profit-margin setting			Low-profit-margin setting		
	A_H	Q^*	Type	A_L	Q^*	Type
0.1	(1299, 1350)	(150, 156)	N	(191, 419)	(75, 150)	Y
0.01	(1060, 1350)	(150, 180)	Y	(105, 365)	(75, 150)	Y
0.001	(118, 1350)	(150, 204)	Y	[225, 450)	(75, 114)	N
0.0001	[0, 1350)	(189, 205)	N	[225, 450)	(75, 78)	N

Table 2: Comparison with the case where $r = 0$ ($D \sim U[0, 300]$, $\lambda = 2$).

	high-profit-margin setting ($q_c = 225$)	low-profit-margin setting ($q_c = 75$)
$\alpha = 0.1$	$q^* = 7$	$q^* = 12$
$\alpha = 0.01$	$q^* = 41$	$q^* = 44$
$\alpha = 0.001$	$q^* = 148$	$q^* = 50$
$\alpha = 0.0001$	$q^* = 204$	$q^* = 48$

Referring to a number of studies in the literature for the coefficient of risk attitude, we use $\alpha = 0.1, 0.01, 0.001$, and 0.0001 . The calculation results are shown in Table 1, from which we can see that large scopes for the reference point lead to the pull-to-center effect. Moreover, we also provide the set of optimal order quantities Q^* corresponding to reference points $r \in A_H(A_L)$ in Table 1. In experiment 1 in [1], the average order quantities in the high- and low-profit-margin settings are 176.68 and 134.06, respectively. Then, from Table 1, as indicated by “Y” type, our model can predict these average order quantities, although the model may fail to predict them in some scopes of behavioral parameters, as indicated by “N” type.

Ignoring the reference point, as in some of the literature, we are interested in whether prospect theory can be used to predict the pull-to-center effect. Then, by setting $r = 0$, we calculate the order quantities, which are shown in Table 2. Clearly, it shows that the results fail to follow the pull-to-center effect. (Only for the high-profit-margin setting, $q^* = 204$ for $\alpha = 0.0001$ can lead to the pull-to-center effect; this is because $r = 0$ has been included in set A_H for $\alpha = 0.0001$ in Table 1.) Consequently, for prospect theory, the reference point plays an important role in predicting the behavior of decision makers.

Table 3: Scope of reference point that leads to the pull-to-center effect ($D \sim U[900, 1200]$, $\lambda = 2$).

α	High-profit-margin setting			Low-profit-margin setting		
	A_H	Q^*	Type	A_L	Q^*	Type
0.1	(9399, 9450)	(1050, 1056)	N	(2891, 3119)	(975, 1050)	Y
0.01	(9160, 9450)	(1050, 1080)	N	(2805, 3065)	(975, 1050)	Y
0.001	(8218, 9450)	(1050, 1104)	Y	[2925, 3150)	(975, 1014)	N
0.0001	[8100, 9450)	(1089, 1105)	Y	[2925, 3150)	(975, 978)	N

In [1], to further examine the use of prospect theory to explain the pull-to-center effect, the authors conducted additional experimental sessions with $D \sim U[900, 1200]$. (The resulting critical fractiles were 0.25 and 0.75 for the two settings and the standard optimal solutions were 975 and 1125, respectively.) The feasible profits lay between 0 and 3600 for the low-profit-margin setting and between 7200 and 10800 for the high-profit-margin setting. That is, for any decision of order quantity in $[900, 1200]$, the profit was guaranteed to be positive. Then, for the case of only gain without loss, by ignoring the reference point, prospect theory degenerated to a risk-averse model. It is known that the order quantity is lower than the standard optimal order quantity q_c in a risk-averse model. However, Schweitzer and Cachon [1] still observed the pull-to-center effect in their experimental data, inconsistent with the predictions of prospect theory. Using a model without a reference point, Nagarajan and Shechter [2] recently provide an analytical proof for the low-profit-margin setting and numerical calculations for the high-profit-margin setting, by which the model predicts a deviation from the center. Hence, they claim that prospect theory cannot predict the pull-to-center effect.

For this additional example, our model yields the opposite results. We use the same coefficients of λ and α as in the previous example for calculation. Again, we can see in Table 3 that considerable scopes of the reference point lead to the pull-to-center effect. In Experiment 2 in [1], the average order quantity in the high-profit-margin setting is 1092.55, and that in the low-profit-margin setting is 1021.81. Then, from Table 3, our model can also predict these average order quantities for considerable scopes of behavioral parameters.

Furthermore, for this additional example, if we ignore the reference point, as in some of the literature, our model degenerates to a risk-averse model. By

Table 4: Comparison with the case where $r = 0$ ($D \sim U[900, 1200]$, $\lambda = 2$).

	high-profit-margin setting ($q_c = 1125$)	low-profit-margin setting ($q_c = 975$)
$\alpha = 0.1$	$q^* = 900$	$q^* = 900$
$\alpha = 0.01$	$q^* = 900$	$q^* = 921$
$\alpha = 0.001$	$q^* = 900$	$q^* = 957$
$\alpha = 0.0001$	$q^* = 900$	$q^* = 973$

setting $r = 0$, we calculate the order quantities, with the results in Table 4. Clearly, instead of predicting the pull-to-center effect, all order quantities are lower than the standard optimal solutions; the model predicts a behavioral tendency in the same way as a risk-averse model does. Again, the results emphasize that for prospect theory, the reference point plays an important role in predicting the behavior of decision makers.

In general, if a decision maker faces a context with higher feasible profit, his/her reference point should also be greater. On the contrary, if a decision maker faces a context with lower (even negative) feasible profit, his/her reference point should be smaller (even negative). Consider the previous example of the low-profit-margin setting with $p = 12$ and $D \sim U[0, 300]$ again. If we set the marginal cost $w = 11$, the feasible profit changes from -3300 to 300 , i.e., it becomes smaller. Then, with $\lambda = 2$ and $\alpha = 0.01$, the calculation of our model yields $A_L = (-8, 150)$, which contains negative reference points, where all reference points can lead to the pull-to-center effect. By similar lines, for other behavioral parameters, e.g., the risk attitude α and the loss aversion λ , a decision maker should have different values of these parameters depending on the context, not fixed over all contexts.

In the end of this section, we provide an additional discussion about the prediction power of prospect theory for the pull-to-center effect. As shown in the previous section, as well as in this section, prospect theory indicates that a newsvendor will exhibit the pull-to-center effect if his/her reference point falls into a certain range (i.e., A_H or A_L). Alternatively, for a newsvendor whose reference point is out of the range A_H or A_L , possibly he/she may fail to follow the pull-to-center effect. In fact, humans are heterogeneous in making decisions, therefore it is not necessary for all newsvendors to follow the pull-to-center effect. Indeed, in many experimental studies, considerable part of subjects do not exhibit the pull-to-center effect in the sense of individual level, although it does from the aggregate level over all subjects (see, e.g., the

analysis in [13]). As a consequence, prospect theory can be a good candidate to explain the decision behavior of newsvendors for predicting either the pull-to-center effect or non-pull-to-center effects.

4. Concluding Remarks

In this paper, we analytically prove that prospect theory with a decision-independent reference point can predict the decision behavior in the newsvendor problem with general distributions of demand. Our model is a typical model of prospect theory, where all probabilities are used without weights, or equivalently, the model uses a linear weighting function for probabilities. A more general model of prospect theory is to use a nonlinear weighting function for probabilities [10, 12]. Obviously, the prediction power of the model with a nonlinear weighting function for probabilities is stronger than the typical model, because the latter is a special one of the former. In this sense, prospect theory (no matter typical model or general model) is powerful in predicting the pull-to-center effect for the problem of newsvendor decision.

A number of experiments have been conducted in the literature, most of which are based on the newsvendor setting with either a uniform or a normal distribution of demand. For different motivations, other distributions can be adopted, and decision-independent reference points when making decisions might be more viable. Our work can be a bridge to connect the experimental study with the model analysis in research on predicting the pull-to-center effect.

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Appendix A. Supplementary Data

All the proofs for Propositions and Theorems related to this article are provided as Online Supplements, which can be found at the journals website.

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Online Supplements on “Revisiting Prospect Theory and the Newsvendor Problem”

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1 Preparatory Materials

For $r \in I_0$, it is not difficult to verify that $q_1(r) \geq \underline{d} \geq q_2(r)$ if and only if $r \in I_1$. Therefore, the expected utility function $V(q|r)$ in problem (1) can be expressed as

$$V(q|r) = \begin{cases} V_0(q|r), & \text{if } r \in I_0 \text{ and } q \in (q_0(r), \bar{d}], \\ V_1(q|r), & \text{if } r \in I_1 \text{ and } q \in [\underline{d}, q_0(r)] \\ V_2(q|r), & \text{if } r \in I_2 \text{ and } q \in [\underline{d}, q_0(r)], \end{cases} \quad (\text{A.1})$$

where

$$\begin{aligned} V_0(q|r) &= \int_{\underline{d}}^{\frac{r+wq}{p}} -\lambda u(r+wq-px)f(x)dx + \int_{\frac{r+wq}{p}}^q u(px-wq-r)f(x)dx + u(pq-wq-r)\bar{F}(q), \\ V_1(q|r) &= \int_{\underline{d}}^q -\lambda u(r+wq-px)f(x)dx - \lambda u(r+wq-pq)\bar{F}(q), \\ V_2(q|r) &= \int_{\underline{d}}^q u(px-wq-r)f(x)dx + u(pq-wq-r)\bar{F}(q). \end{aligned}$$

Lemma 1. *For any given $r \in I_0$: (1) if $r \in I_2$, or $r \in I_1$ and $\lambda = 1$, then $V(q|r)$ is continuously differentiable with respect to $q \in [\underline{d}, \bar{d}]$; (2) if $r \in I_1$ and $\lambda > 1$, then $V(q|r)$ is continuously differentiable with respect to $q \in [\underline{d}, q_1(r)) \cup (q_1(r), \bar{d}]$, and is continuous at point $q = q_1(r)$ but not differentiable at this point.*

This lemma can be directly proved from (A.1), hence we omit the details here. This lemma lays the foundation for all subsequent results.

Lemma 2. *For any given $r \in I_0$, if $p > 2w$, $Q_0(r)$ is a singleton or empty set.*

Proof. From (A.1), we have $V(q|r) = V_0(q|r)$ for $q > q_0(r)$. Noting that any $q \in Q_0(r)$ must satisfy $dV(q|r)/dq = 0$, we only need to show that there is at most one $q \in (q_0(r), \bar{d}]$ satisfying $dV_0(q|r)/dq = 0$. Define

$$G_r(q) = \frac{1}{\alpha e^{\alpha(r-wq)}} \frac{dV_0(q|r)}{dq}.$$

Then we only need to prove that $G_r(q) = 0$ has at most one root. For this purpose, we calculate the first derivative of $G_r(q)$ as

$$\begin{aligned} \frac{dG_r(q)}{dq} &= -2\alpha w^2 \int_{\frac{r+wq}{p}}^q e^{\alpha(2wq-px)} f(x) dx - \alpha(p-w)(p-2w)e^{\alpha(2wq-pq)} \bar{F}(q) \\ &\quad - (\lambda-1) \frac{w^2}{p} e^{\alpha(wq-r)} f\left(\frac{r+wq}{p}\right) - pe^{\alpha(2wq-pq)} f(q). \end{aligned}$$

Noting that all the items in the righthand side are nonpositive and cannot be all zeros, we have $dG_r(q)/dq < 0$. That is, $G_r(q)$ is strictly decreasing on $(q_0(r), \bar{d}]$, so it has at most one zero point. ■

Lemma 3. For $r \in I_0$, if $f'(x) \geq 0$ on $[\underline{d}, F_{0.5}]$ and $q_0(r) < F_{0.5}$, $V(q|r)$ is strictly concave with respect to $q \in (q_0(r), F_{0.5}]$.

Proof. From (A.1), we have $V(q|r) = V_0(q|r)$ for $q \in (q_0(r), F_{0.5}]$. We only need to verify $d^2V_0(q|r)/dq^2 < 0$. In fact,

$$\begin{aligned} \frac{d^2V_0(q|r)}{dq^2} &= (1-\lambda) \frac{w^2}{p} u'(0) f\left(\frac{r+wq}{p}\right) - pu'(pq-wq-r)f(q) - \lambda w^2 \int_{\underline{d}}^{\frac{r+wq}{p}} u''(r+wq-px)f(x) dx \\ &\quad + w^2 \int_{\frac{r+wq}{p}}^q u''(px-wq-r)f(x) dx + (p-w)^2 u''(pq-wq-r) \bar{F}(q) \\ &= \frac{w^2-p^2}{p} u'(pq-wq-r)f(q) - \lambda \frac{w^2}{p} u'(r+wq-p\underline{d})f(\underline{d}) - \lambda \frac{w^2}{p} \int_{\underline{d}}^{\frac{r+wq}{p}} u'(r+wq-px)f'(x) dx \\ &\quad - \frac{w^2}{p} \int_{\frac{r+wq}{p}}^q u'(px-wq-r)f'(x) dx + (p-w)^2 u''(pq-wq-r) \bar{F}(q) \\ &< 0. \end{aligned}$$

The last inequality follows from the facts that $u' > 0$, $u'' < 0$ and $f'(x) \geq 0$ on $[\underline{d}, F_{0.5}]$. ■

2 Proofs of the Main Results

Proof of Proposition 1. It is difficult to complete the proof for the proposition by usual calculus. In stead, we first show that the result holds for any stepwise cumulative distribution function, and then a limiting argument is used to show that it also holds for general cumulative distribution functions. The proof needs to be proceed in three stages as follows, and the detailed procedure can be referred to [?].

Stage 1: For any given $r \in I_0$, if $q > q_0(r)$ and $q \notin \Omega$, $V(q|r)$ is differentiable at q , where $\Omega = \{\omega_1, \omega_2, \dots\}$ is the set of all the discontinuity points of any given stepwise cumulative distribution function.

Stage 2: For any given $r \in I_0$, $dV(q|r)/dq$ is continuous at any $q \in [q_0(r), \bar{d}] \cap \Omega^c$, and for any $q_n \in [q_0(r), \bar{d}] \cap \Omega$, $\lim_{q \rightarrow q_n^-} dV(q|r)/dq \leq \lim_{q \rightarrow q_n^+} dV(q|r)/dq$.

Stage 3: For $i = 1, 2, \dots$, one and only one of the followings holds:

- (1) $dV(q|r)/dq > 0$ on $(\omega_i, \omega_{i+1}) \cap [q_0(r), \bar{d}]$,
- (2) $dV(q|r)/dq < 0$ on $(\omega_i, \omega_{i+1}) \cap [q_0(r), \bar{d}]$,
- (3) there exists a $\theta \in (\omega_i, \omega_{i+1}) \cap [q_0(r), \bar{d}]$ such that $dV(\theta|r)/dq = 0$, $dV(q|r)/dq > 0$ on (ω_i, θ) and $dV(q|r)/dq < 0$ on (θ, ω_{i+1}) . ■

Proof of Proposition 2. We consider the following two subproblems to solve problem (1) :

$$\max_{q \in [q_0(r), \bar{d}]} V(q|r), \quad (\text{A.2})$$

and

$$\max_{q \in [\underline{d}, q_0(r)]} V(q|r). \quad (\text{A.3})$$

The solution sets of subproblems (A.2) and (A.3) are denoted by $Q_1^*(r)$ and $Q_2^*(r)$ (Q_1^* and Q_2^* for short), respectively. It is obvious that $Q^*(r) \subseteq Q_1^* \cup Q_2^*$.

For subproblem (A.2), the optimum must be the boundary point $q_0(r)$ or \bar{d} , or contained in $Q_0(r)$. Noting that $dV(\bar{d}|r)/dq < 0$, we know that \bar{d} cannot be an optimum. Therefore, $Q_1^* \subseteq Q_0(r) \cup \{q_0(r)\}$.

Now we turn to subproblem (A.3). First, we consider the case $r \in I_1$, where $V(q|r) = V_1(q|r)$ when $q \in [\underline{d}, q_0(r)]$. Note that, for $q \in [\underline{d}, \min(q_c, q_0(r))]$,

$$\begin{aligned} \frac{dV_1(q|r)}{dq} &> \lambda u'(r + wq - pq)(-wF(q) + (p - w)\bar{F}(q)) \\ &= \lambda u'(r + wq - pq)((p - w) - pF(q)) \\ &\geq \lambda u'(r + wq - pq)((p - w) - pF(q_c)) \\ &= 0. \end{aligned}$$

This implies that $V(q|r)$ is strictly increasing in $q \in [\underline{d}, \min(q_c, q_0(r))]$.

If $r \in [(p\underline{d} - w\underline{d}, (p - w)q_c]$, then $q_0(r) \leq q_c$. Therefore, we have $Q_2^* = \{q_0(r)\}$, which implies $Q^*(r) \subseteq Q_1^* \cup Q_2^* \subseteq Q_0(r) \cup \{q_0(r)\}$.

If $r \in ((p - w)q_c, (p - w)\bar{d}]$, then $q_0(r) > q_c$. Therefore, any $q \in [\underline{d}, q_c]$ is not an optimum for $V(q|r)$ under the constraint $q \in [\underline{d}, q_0(r)]$. Thus, Q_2^* should be a subset of $(q_c, q_0(r)]$, which implies $Q^*(r) \subseteq Q_1^* \cup Q_2^* \subseteq Q_0(r) \cup (q_c, q_0(r)]$.

Next, we consider the case $r \in I_2$. Since $V(q|r) = V_2(q|r)$ when $q \in [\underline{d}, q_0(r)]$, subproblem (A.3) is equivalent to $\max_{\underline{d} \leq q \leq q_0(r)} V_2(q|r)$. We can claim that $V_2(q|r)$ is a concave function with respect to q because

$$\begin{aligned} \frac{d^2 V_2(q|r)}{dq^2} &= w^2 \int_{\underline{d}}^q u''(px - wq - r)f(x)dx \\ &\quad + (p - w)^2 u''(pq - wq - r)\bar{F}(q) - pu'(pq - wq - r)f(q) \\ &< 0. \end{aligned}$$

The inequality follows from the facts $u' > 0$ and $u'' < 0$.

Note that

$$\left. \frac{dV_2(q|r)}{dq} \right|_{q=\underline{d}} = (p - w)u'(p\underline{d} - w\underline{d} - r) > 0.$$

If $r \in I_2 \setminus B$,

$$\left. \frac{dV_2(q|r)}{dq} \right|_{q=q_0(r)} \geq 0.$$

From the concavity of $V_2(q|r)$, $V_2(q|r)$ is increasing in $q \in [\underline{d}, q_0(r)]$, which implies $Q_2^* = \{q_0(r)\}$. Therefore $Q^*(r) \subseteq Q_1^* \cup Q_2^* \subseteq Q_0(r) \cup \{q_0(r)\}$.

If $r \in B$,

$$\left. \frac{dV_2(q|r)}{dq} \right|_{q=q_0(r)} < 0.$$

From the concavity of $V_2(q|r)$, $Q_2^* = \{q^*\}$ is a singleton set and q^* is the solution of $M(q) = dV_2(q|r)/dq = 0$. Because $(dV(q|r)/dq)|_{q=q_0(r)} = (dV_2(q|r)/dq)|_{q=q_0(r)} < 0$, $V(q|r)$ is decreasing in $q \geq q_0(r)$ by Proposition 1. Therefore, $Q_1^* = \{q_0(r)\}$, and problem (1) degenerates to subproblem (A.3), which means $Q^*(r) = Q_2^* = \{q^*\}$. ■

Proof of Theorem 1. Since $p > 2w$, we only need to prove that the set A_H constructed in Algorithm 1 works for this theorem. First, we show that $dV(q_c|r)/dq < 0$ and $dV(F_{0.5}|r)/dq > 0$ for $r \in A_H$. Noting that $H(q, r) = dV_0(q|r)/dq$ which is a continuous function, we only need to prove $H(q_c, r) < 0$ and $H(F_{0.5}, r) > 0$ for $r \in A_H$, since $q_0(r) < F_{0.5} < q_c$ at this situation.

Because $\lambda \geq 1$, $(p-2w)q_c + px - 2(p-w)\underline{d} > (p-2w)(q_c - \underline{d}) \geq 0$ for $x \geq \underline{d}$, and $p(q_c - x) > 0$ for $x < q_c$, we have

$$\begin{aligned} H(q_c, (p-w)\underline{d}) &= \alpha e^{-\alpha(p-w)(q_c-\underline{d})} \left[-\lambda w \int_{\underline{d}}^{\frac{wq_c+(p-w)\underline{d}}{p}} e^{\alpha((p-2w)q_c+px-2(p-w)\underline{d})} f(x) dx \right. \\ &\quad \left. - w \int_{\frac{wq_c+(p-w)\underline{d}}{p}}^{q_c} e^{\alpha p(q_c-x)} f(x) dx + (p-w)\bar{F}(q_c) \right] \\ &< \alpha e^{-\alpha(p-w)(q_c-\underline{d})} [-wF(q_c) + (p-w)\bar{F}(q_c)] \\ &= 0. \end{aligned}$$

So we must have $H(q_c, r) < 0$ for any $r \in [(p-w)\underline{d}, \min(r_1, (p-w)F_{0.5}))$ from the definition of r_1 . For the case $H(F_{0.5}, (p-w)\underline{d}) > 0$, $H(F_{0.5}, r) > 0$ is also guaranteed for $r \in [(p-w)\underline{d}, \min(r_2, (p-w)F_{0.5}))$. Then, for $r \in A_H = [(p-w)\underline{d}, \min(r_1, r_2, (p-w)F_{0.5}))$, $dV(q_c|r)/dq = H(q_c, r) < 0$ as well as $dV(F_{0.5}|r)/dq = H(F_{0.5}, r) > 0$.

If $H(F_{0.5}, (p-w)\underline{d}) \leq 0$, because $\lambda \leq (p-w)/w$ and $pF_{0.5} - px > 0$ for $x < F_{0.5}$, it follows that

$$\begin{aligned} H(F_{0.5}, (p-w)F_{0.5}) &= \alpha \left[-\lambda w \int_{\underline{d}}^{F_{0.5}} e^{-\alpha(pF_{0.5}-px)} f(x) dx + (p-w)\bar{F}(F_{0.5}) \right] \\ &> \alpha [-0.5\lambda w + 0.5(p-w)] \\ &\geq 0. \end{aligned}$$

So we know that set $\{r|r \in [(p-w)\underline{d}, (p-w)F_{0.5}] \text{ and } H(F_{0.5}, r) = 0\}$ is nonempty and $r_3 < (p-w)F_{0.5}$ from the continuity of H with respect to r . The definition of r_3 also tells the fact that $H(F_{0.5}, r) > 0$ for $r \in (r_3, (p-w)F_{0.5}]$. Noting that $dV(F_{0.5}|r_3)/dq = H(F_{0.5}, r_3) = 0$, and Lemma 2 tells that $Q_0(r_3)$ has at most one element, we must have $Q_0(r_3) = \{F_{0.5}\}$. Since $q_c > F_{0.5}$, $H(q_c, r_3) = dV(q_c|r_3)/dq < 0$ by Proposition 1. Since $r_4 = \inf\{r|r \in [r_3, (p-w)F_{0.5}] \text{ and } H(q_c, r) = 0\}$, we must have $H(q_c, r) < 0$ for $r \in A_H = (r_3, \min((p-w)F_{0.5}, r_4))$. Therefore, $H(q_c, r) < 0$ and $H(F_{0.5}, r) > 0$ for $r \in A_H$.

We further prove that $Q^*(r) \subset (F_{0.5}, q_c)$ for all $r \in A_H \subseteq [(p-w)\underline{d}, (p-w)F_{0.5}]$. Since for $r \in A_H$, $dV(q_c|r)/dq = H(q_c, r) < 0$ and $dV(F_{0.5}|r)/dq = H(F_{0.5}, r) > 0$, we know that $Q_0(r)$ is a nonempty subset of $(F_{0.5}, q_c)$. In addition, Lemma 2 implies that $Q_0(r)$ is a singleton set. According to Proposition 2, $Q^*(r) \subseteq Q_0(r) \cup \{q_0(r)\}$. Note that $dV(F_{0.5}|r)/dq > 0$ and Proposition 1 implies that $V(F_{0.5}|r) > V(q_0(r)|r)$, we know $q_0(r)$ is not the local optimum. Consequently, $Q^*(r) = Q_0(r) \subset (F_{0.5}, q_c)$ is a singleton set. ■

Proof of Theorem 2. Since $p < 2w$, we only need to prove that the set A_L constructed in Algorithm 1 works for this theorem. We will show that $Q^*(r) \subseteq (q_c, F_{0.5})$ for $r \in A_L$.

Because $-\alpha p x > -\alpha p q_c$ for $x < q_c$, it follows that

$$\begin{aligned} H(q_c, p\underline{d} - wq_c) &= \alpha e^{-\alpha p(q_c - \underline{d})} \left[-w \int_{\underline{d}}^{q_c} e^{\alpha p(q_c - x)} f(x) dx + (p - w)\overline{F}(q_c) \right] \\ &< \alpha e^{-\alpha p(q_c - \underline{d})} (-wF(q_c) + (p - w)\overline{F}(q_c)) \\ &= 0. \end{aligned}$$

First, consider the case $H(q_c, (p - w)q_c) > 0$. For this case, the set $\{r | r \in (p\underline{d} - wq_c, (p - w)q_c) \text{ and } H(q_c, r) = 0\}$ is nonempty. It is clear that $H(q_c, r) > 0$ for $r \in (r_5, (p - w)q_c)$ and $H(q_c, r_5) = 0$. In addition, Lemma 3 implies that $H(q, r)$ is strictly decreasing in $q \in [q_0(r), F_{0.5}]$ if $r \in I_0$ and $f'(x) \geq 0$ for $x \in [\underline{d}, F_{0.5}]$. Hence $H(F_{0.5}, r_5) < H(q_c, r_5) = 0$. Then for $r \in A_L = (r_5, \min(r_6, (p - w)F_{0.5}))$, we can claim $H(F_{0.5}, r) < 0$.

If $r \in A_L \cap (r_5, (p - w)q_c)$, we can observe that $Q^*(r) \subseteq Q_0(r) \cup \{q_0(r)\}$. In addition, we have $q_0(r) < q_c < F_{0.5}$, so $dV(q_c|r)/dq = dV_0(q_c|r)/dq = H(q_c, r) > 0$ as well as $dV(F_{0.5}|r)/dq = dV_0(F_{0.5}|r)/dq = H(F_{0.5}, r) < 0$, thus we can claim that $Q_0(r) \subseteq (q_c, F_{0.5})$. Noting that $dV(q_c|r)/dq > 0$ also implies that V strictly increases near point $q = q_c$, we have $V(q_c|r) > V(q_0(r)|r)$ by Proposition 1. Hence, $q_0(r)$ is not the local optimum, and $Q^*(r) \subseteq Q_0(r) \subseteq (q_c, F_{0.5})$.

If $r \in A_L \cap [(p - w)q_c, (p - w)F_{0.5})$, we know $q_c \leq q_0(r) < F_{0.5}$ and $dV(F_{0.5}|r)/dq = dV_0(F_{0.5}|r)/dq = H(F_{0.5}, r) < 0$. By Proposition 1, we obtain $V(q|r) \leq V(F_{0.5}|r)$ for $q > F_{0.5}$, and thus $Q^*(r) \subseteq (\underline{d}, F_{0.5}]$. Therefore, Proposition 2 implies $Q^*(r) \subseteq ((q_c, q_0(r)] \cup Q_0(r)) \cap (\underline{d}, F_{0.5}] \subseteq (q_c, F_{0.5})$.

Now turn to the case $H(q_c, (p - w)q_c) \leq 0$. For this case, we know $F_{0.5} > q_c = q_0(pq_c - wq_c)$, and then $H(F_{0.5}, (p - w)q_c) < H(q_c, (p - w)q_c) \leq 0$ by the fact that $H(q, r)$ is strictly decreasing in $q \in [q_0(pq_c - wq_c), F_{0.5}]$ if $f'(x) \geq 0$ for $x \in [\underline{d}, F_{0.5}]$. Then we must have $H(F_{0.5}, r) < 0$ for each $r \in A_L = [(p - w)q_c, \min(r_7, (p - w)F_{0.5}))$. Note that $q_0(r) < F_{0.5}$ for $r \in A_L$, $dV(F_{0.5}|r)/dq = dV_0(F_{0.5}|r)/dq = H(F_{0.5}, r) < 0$. It follows from Proposition 1 that $V(q|r)$ is decreasing in $q > F_{0.5}$. Hence, the local optimum of problem (1) is less than $F_{0.5}$, i.e., $Q^*(r) \subseteq (\underline{d}, F_{0.5})$. According to Proposition 2, $Q^*(r) \subseteq (q_c, q_0(r)] \cup Q_0(r)$. Therefore, we have $Q^*(r) \subseteq ((q_c, q_0(r)] \cup Q_0(r)) \cap (\underline{d}, F_{0.5}) \subseteq (q_c, F_{0.5})$. ■

References

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