# Asymptotic Optimality of Base-Stock Policies for Lost-Sales Inventory Systems with Stochastic Lead Times

Shilin Yuan[a], Jiameng Lyu[a], Jinxing Xie[a], Yuan Zhou[b,c]

[a]Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
[b]Yau Mathematical Sciences Center & Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
[c]Beijing Institute of Mathematical Sciences and Application, Beijing, 101408, China

**Abstract**

We consider the lost-sales inventory systems with stochastic lead times and establish the asymptotic optimality of base-stock policies for such systems. Specifically, we prove that as the per-unit lost-sales penalty cost becomes large compared to the per-unit holding cost, the ratio of the optimal base-stock policy's cost to the optimal cost converges to one. Our paper provides a theoretical guarantee of the widely adopted base-stock policies in lost-sales inventory systems with stochastic lead times for the first time.

*Keywords:* lost-sales inventory system, stochastic lead time, asymptotic analysis, base-stock policy

## 1. Introduction

The management of lost-sales inventory systems with positive lead times, as a fundamental problem in inventory management, has received tremendous attention from both academia and industries in the past several decades. Since the optimal policy for such inventory systems is computationally intractable due to the notorious curse of dimensionality, investigating the performance of simple implementable policy classes becomes quite important in practice. Among them, the class of base-stock policies is one of the most widely adopted heuristics in practice for its simplicity and convenience to implement.

The performance of base-stock policies in lost-sales inventory systems with *constant* lead times has been well studied both numerically and theoretically. For example, Zipkin [22] observed base-stock policies perform reasonably well in a variety of numerical experiments; Huh et al. [14] proved that the base-stock policies are asymptotically optimal as lost-sales cost becomes large compared with the holding cost, which is common in many business scenarios.

In recent years, the uncertainty in lead times has become more and more prevailing especially during and after the COVID-19 pandemic, due to factors such as transportation delays, supply stockouts, port congestion, and so on. In such instances, the lead time should be treated as a random variable instead of a constant. However, the theoretical performance of base-stock policies for lost-sales inventory systems with *stochastic* lead times has not been investigated in the literature.

In this paper, we study the lost-sales inventory systems with stochastic lead times and establish the first asymptotic optimality result for base-stock policies under stochastic lead times. Specifically, we consider the regime that per-unit holding cost $h$ is fixed and per-unit lost-sales cost $b$ approaches infinity. The optimal cost of the problem (denoted by $C^{\mathcal{L},*}(h,b)$) and the cost of the optimal base-stock policy (denoted by $\inf_{S\geq 0} C^{\mathcal{L},S}(h,b)$, where $S$ is the parameter of the base-stock policy) are asymptotically equal, that is

$$\lim_{b\to+\infty} \frac{\inf_{S\geq 0} C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L},*}(h,b)} = 1.$$

The proof follows the roadmap of [14] that relates the costs of the lost-sales system to newsvendor costs, but the technical details are different from it. Specifically, to establish the lower bound on the optimal cost, we first generalize the imitation argument in [15] to a stochastic lead time version. Then, we adopt the sample-path analysis from [5] to the problem with stochastic lead times and establish the upper bound of cost under base-stock policies. The sample-path analysis avoids proving comparison results by the existence of the steady-state distribution as in [14], which seems to not work under stochastic lead times.

We would like to remark that although we generalize the existing methods to tackle the difficulties caused by stochastic lead times, we do not claim it as our core technical contribution. Our main contribution lies in establishing the asymptotic optimality of the widely adopted base-stock policies in lost-sales inventory systems with stochastic lead times for the first time. We also present a useful result (Lemma 2) to establish the sublinear mean residual life property, which may be of independent interest for later research about stochastic lead times. By Lemma 2, we show our result holds under many demand distributions. Finally, we also discuss the limitation of existing methods to tackle the other lead time models; see Section 7. We think these unique challenges caused by stochastic lead times would be an interesting future direction.

*Email addresses:* yuansl21@mails.tsinghua.edu.cn (Shilin Yuan), lvjm21@mails.tsinghua.edu.cn (Jiameng Lyu), xiejx@tsinghua.edu.cn (Jinxing Xie), yuan-zhou@tsinghua.edu.cn (Yuan Zhou)

## 2. Brief Literature Review

This section briefly reviews the most related literature on inventory systems with stochastic lead times and lost-sales inventory systems with positive lead times. Please refer to [4] for a broad review of lost-sales inventory systems and [12] for a detailed review of the asymptotic analysis of various inventory systems.

For lost-sales inventory systems with deterministic lead times, Zipkin [22] tested various heuristics and found that base-stock policies and constant order policies perform reasonably well. Huh et al. [14] established the asymptotic optimality of base-stock policies in the constant lead times setting, when the ratio of holding and lost-sales cost approaches infinite. Bijvank et al. [3] demonstrated the robustness of the asymptotic optimality results in [14]. Bu et al. [5] established the asymptotic optimality of base-stock policies in three classes of inventory systems, including the partial backlogged systems with deterministic lead times. There is a series of works studying the constant order policies. Goldberg et al. [11] proved the asymptotic optimality result in the regime of large lead times and Xin and Goldberg [19] proved the optimality gap converges to zero exponentially fast. Later, many policies based on constant-order policies are proved to be asymptotically optimal for various systems [7, 2, 1]. Xin [18] considered capped base-stock policies, which enjoy the advantages of both base-stock policies and constant order policies.

## 3. Problem Formulation

In this paper, we consider a lost-sales inventory system with stochastic lead times. The periods are indexed by $t = 1, 2, \dots,$ and demand in period $t$ is denoted by $D_t$. The demand is independent over time and identically distributed with common distribution $D$. We further assume that $\mathbb{E}[D] > 0$ to avoid triviality. The lead time in period $t$ is $L_t \in \mathbb{N}$, i.e., the order placed in period $t$ will arrive in period $t + L_t$. We use $\{L_1, L_2, \dots\}$ to denote the lead time process and require that $L_t \le \overline{L}, \forall t \ge 1$ for some non-negative upper bound $\overline{L}$.

Now we state the sequence of events as follows,

1. At the beginning of period $t$, the firm reviews its on-hand inventory level $IL_t$ and inventories in the pipeline $(q_{t-\overline{L}}, q_{t-\overline{L}+1}, \dots, q_{t-1})$, where $q_i$ is the order placed at period $i$. The inventory position (i.e., the on-hand inventory plus all inventory in delivery) is
$$IP_t = IL_t + \sum_{\{i:\ i+L_i \ge t\}} q_i.$$
We assume the initial inventory states $(q_{1-\overline{L}}, q_{1-\overline{L}+1}, \dots, q_0) = (0, 0, \dots, 0)$ and $IP_1 = 0$.
2. The firm decides its order quantity $q_t \ge 0$ and receives all orders that should arrive at period $t$. The total inventory received is
$$Q_t := \sum_{\{i:\ i+L_i = t\}} q_i.$$
After the delivery, the on-hand inventory becomes $I_t = IL_t + Q_t$.
3. Demand is realized as $D_t$ and is satisfied to the maximal extent using on-hand inventory, and unsatisfied demand is lost. The leftover inventory causes a holding cost of $h$ per unit and unsatisfied demand causes a lost-sales cost of $b$ per unit. Therefore, the total cost incurred at period $t$ is
$$C_t = h(I_t - D_t)^+ + b(D_t - I_t)^+.$$

A policy $\pi = (\pi_1, \pi_2, \dots)$ is admissible if for each period $t \ge 1$, $\pi_t$ maps the inventory state to $q_t \ge 0$ and it is measurable. Let $\Pi$ be the set of all admissible policies. We use $I_t^\pi$ to denote the after-delivery on-hand inventory under the admissible policy $\pi$, and the cost in period $t$ is
$$C_t^\pi = h(I_t^\pi - D_t)^+ + b(D_t - I_t^\pi)^+.$$

The firm aims to find an optimal policy in the set $\Pi$ of admissible policies to minimize the long-run average expected inventory cost as follows
$$\inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[C_t^\pi].$$

In this paper, we focus on the performance of the class of base-stock policies. For $S \ge 0$, the base-stock policy with parameter $S$ (or order-up-to-$S$ policy) decides order quantity $q_t$ to raise its inventory position up to $S$ in each period $t \ge 1$, that is
$$q_t = (S - IP_t)^+.$$

We use $C^{\mathcal{L},*}(h, b)$ and $C^{\mathcal{L},S}(h, b)$ to denote the long-run average cost of the lost-sales system under the optimal cost and order-up-to-$S$ policy, respectively.

Next, we make some assumptions on the lead time process and demand distribution throughout this paper.

**Assumption 1.** *Throughout this paper, we require the lead time process to satisfy the following assumptions.*

1. ***Independence with demand.*** *The lead time process $\{L_t\}_{t\ge1}$ is independent of demand process $\{D_t\}_{t\ge1}$.*
2. ***Non-crossing.*** *The lead time process $\{L_t\}_{t\ge1}$ is non-crossing, i.e., for any $t_1 \le t_2$, $t_1 + L_{t_1} \le t_2 + L_{t_2}$ w.p.1,*
3. ***Stationary.*** *The process $\{(L_t, L_{t+1}, \dots, L_{t+\overline{L}})\}_{t\ge1}$ is a multi-dimentional stationary process.*

Assumption 1.1 is natural because the supply and demand sides are highly independent parts of supply chains. For Assumption 1.2, generally, the order placed earlier should arrive earlier, resulting in non-crossing lead times. Assumption 1.3 allows the condition of supply to be stochastically fluctuating, but its distribution should be stationary.

We define a new process $\{\lambda_1, \lambda_2, \dots\}$ based on the lead time process $\{L_1, L_2, \dots\}$, where $\lambda_t := \max\{i : i + L_i \le t\}$ is the index of the period that has the latest shipment in or before period $t$.

By the definition of $\lambda_{t+1}$, we know

$$\begin{aligned}\lambda_{t+1} &= \max\{i : i + L_i \le t+1\} \\ &= \max\{i+1 : i + L_{i+1} \le t\} \\ &= 1 + \max\{i : i + L_{i+1} \le t\}.\end{aligned}$$

Moreover, we know that $L_t \le \overline{L}$ and $\lambda_t$ only depends on $(L_{t-\overline{L}}, \dots, L_t)$, which is stationary. It follows that $\lambda_{t+1} = 1 + \max\{i : i + L_{i+1} \le t\}$ and $1 + \lambda_t$ have the same distribution. Therefore, $\{t - \lambda_t\}_{t \ge 1}$ follow the same distribution and we define

$$\mathcal{D}_t = \sum_{i=\lambda_t}^{t} D_i, t \ge 1, \tag{1}$$

which is identically distributed by the above discussion. Denote the common distribution and its cumulative distribution function by $\mathcal{D}$ and $F_{\mathcal{D}}(x)$, respectively.

We define the mean residual life of a random variable $X$ as

$$m_X(x) = \begin{cases} \mathbb{E}[X - x \mid X > x] & \text{, if } x < \sup\{x : F_X(x) < 1\}, \\ 0 & \text{, otherwise,} \end{cases}$$

where $F_X(x)$ is the cumulative distribution function of $X$. We assume $\mathcal{D}$ has sublinear mean residual life as follows.

**Assumption 2.** *$m_{\mathcal{D}}(x)$ satisfies that*

$$\lim_{x \to \infty} \frac{m_{\mathcal{D}}(x)}{x} = 0.$$

*For convenience, we also write $m_{\mathcal{D}}(x) = o(x)$ for short.*

The above assumption that $\mathcal{D}$ has sublinear mean residual life is a common assumption used in asymptotic analysis. Note the random summation form of $\mathcal{D}$ (Eq. (1)) causes unique challenges to the verification of Assumption 2. In section 4.1, we will discuss how to verify the assumption for $\mathcal{D}$ and show that many demand distributions satisfy it.

## 4. Discussion on Assumptions

In this section, we discuss assumptions on the lead time process and demand. In Section 4.1, we show a wide range of demand distributions satisfying Assumption 2. In Section 4.2, we give common lead time models that satisfy Assumption 1.

### *4.1. Demand Distributions*

The sublinear mean residual life property (Assumption 2) is significant in the asymptotic analysis of inventory systems with deterministic lead times. For example, see [14, 5, 8, 6]. In this section, we prove a useful lemma for establishing the property for the problem with stochastic lead times.

The following lemma proved by [14] is useful in showing that one-period demand $D$ has sublinear mean residual life.

**Lemma 1.** *If any of the following conditions holds, then $D$ satisfies that $m_D(x) = o(x)$.*

1. *$D$ is bounded.*
2. *$D$ has an Increasing Failure Rate (IFR) distribution.*
3. *$D$ has a finite variance and the distribution $F$ of $D$ has a density $f$ and a failure rate function $r(t)$ of $F$ that satisfies*
$$\lim_{t \to \infty} t \cdot r(t) = \infty,$$
*where for any $t \ge 0$, $r(t) = f(t)/(1 - F(t))$.*

Note the boundedness and IFR property are preserved under deterministic summation. Faced with deterministic lead times, the summation of Eq. (1) is deterministic, and showing the one-period demand to be bounded or have IFR distribution is enough to verify Assumption 2 for many distributions.

However, the stochastic lead times cause a complicated random summation form of $\mathcal{D}$. Both items 2 and 3 in Lemma 1 seem not preserved under random summation, and item 1 excludes a wide range of distributions. Therefore, Lemma 1 is not enough for the systems with stochastic lead times.

Fortunately, we show that if the sublinear mean residual life property is preserved under deterministic summation, it is also preserved under independent random summation.

**Lemma 2.** *Suppose that non-negative random variables $X_1$, $X_2, \dots, X_{\overline{L}}$ satisfy that $m_{X_{1:k}}(x) = o(x)$, for $k \in [\overline{L}]$, where we define $X_{1:k} = X_1 + X_2 + \dots + X_k$. Let $N$ be a random variable supported on $\{0, 1, 2, \dots, \overline{L}\}$ and independent of $X_1, X_2, \dots, X_{\overline{L}}$. Define $Y = \sum_{i=1}^{N} X_i$, then we have*

$$\lim_{x \to \infty} \frac{m_Y(x)}{x} = 0.$$

Applying the above conclusion, we obtain the following corollary, which shows Assumption 2 holds for a large range of common distributions.

**Lemma 3.** *If the one-period demand $D$ is bounded or has an IFR distribution, then $\mathcal{D}$ satisfies Assumption 2. Moreover, if $D$ has one of the following distributions, Assumption 2 holds.*

1. *Discrete (or continuous) uniform, binomial, and hypergeometric distributions.*
2. *Geometric, Poisson, negative binomial (with $r > 0$ and $0 < p < 1$), exponential, and Gaussian distributions.*

*Proof.* Take $Y = \mathcal{D}$, $N = t - \lambda_t + 1$, and $X_i = D_i, i \in [\overline{L}]$. If $D$ is bounded or has IFR distribution, for $k \in [\overline{L}]$, $X_{1:k}$ is also bounded or has IFR distribution, respectively. By Lemma 1, we know that $m_{X_{1:k}}(x) = o(x)$ for $k \in [\overline{L}]$. Therefore, by Lemma 2, we know that $m_{\mathcal{D}}(x) = m_Y(x) = o(x)$, i.e., $\mathcal{D}$ satisfies Assumption 2.

For the distributions in Item 1, it is easy to see that they are bounded distributions. For the distributions listed in Item 2, we can verify that they have IFR distributions. Please refer to [14] for verification. □

Now we give the *Proof of Lemma 2*.

*Proof.* By the definition of $m_Y(x)$, we know

$$\frac{m_Y(x)}{x} = \frac{\int_x^\infty \mathbb{P}[Y > u]\mathrm{d}u}{x\mathbb{P}[Y > x]}.$$

Define $p_k = \mathbb{P}[N = k]$, we have $\mathbb{P}[Y > u] = \sum_{k=1}^{\overline{L}} p_k \cdot \mathbb{P}[Y > u \mid N = k]$ and

$$\mathbb{P}[Y > u \mid N = k] = \mathbb{P}\left[\sum_{i=1}^{k} X_i > u\right].$$

By the above equations, we obtain

$$\frac{m_Y(x)}{x} \leq \sum_{k=1}^{\overline{L}} \frac{p_k \int_x^\infty \mathbb{P}[\sum_{i=1}^k X_i > u]\mathrm{d}u}{x\mathbb{P}[Y > x]}.$$

It suffices to estimate each term on the above right-hand side. If $p_k = 0$, the desired result is trivial. Consider each $k \in [\overline{L}]$ and $p_k > 0$. We have

$$\begin{aligned}
&\lim_{x\to\infty} \frac{p_k \int_x^\infty \mathbb{P}[\sum_{i=1}^k X_i > u]\mathrm{d}u}{x\mathbb{P}[Y > x]} \\
&\leq \lim_{x\to\infty} \frac{p_k \int_x^\infty \mathbb{P}[\sum_{i=1}^k X_i > u]\mathrm{d}u}{x\sum_{k=1}^{\overline{L}} p_k \mathbb{P}[\sum_{j=1}^k X_j > x]} \\
&\leq \lim_{x\to\infty} \frac{p_k \int_x^\infty \mathbb{P}[\sum_{i=1}^k X_i > u]\mathrm{d}u}{x p_k \mathbb{P}[\sum_{i=1}^k X_i > x]} = \lim_{x\to\infty} \frac{m_{X_{1:k}}(x)}{x} = 0,
\end{aligned}$$

where we define $X_{1:k} = X_1 + X_2 + \cdots + X_k$ and the last equality is by that $m_{X_{1:k}}(x) = o(x)$ for any $k \in [\overline{L}]$ and we complete the proof. □

### 4.2. Lead Time Processes

First, we give some representative lead time models. As we will see, the conditions of Assumption 1 can be verified by the definition. We will see that base-stock policies are asymptotically optimal under these lead time models.

**Kaplan's Lead Time Model.** $\{L_t\}_{t\geq 1}$ is determined by *i.i.d.* random variables $\{\rho_t\}_{t\geq 1}$ with common distribution $\rho$ and $\rho \leq \overline{L}$. Specifically, in period $t$, all orders that have been outstanding for at least $\rho_t$ periods arrive immediately.

Note the order placed earlier stays outside longer, thus it arrives earlier. We know $\{L_t\}_{t\geq 1}$ is non-crossing. By the definition, $\{L_t = n\} = \{\rho_t > 0, \rho_{t+1} > 1, \ldots, \rho_{t+n-1} > n-1, \rho_{t+n} \leq n\}$. Therefore, the process $\{(L_t, L_{t+1}, \ldots, L_{t+\overline{L}})\}_{t\geq 1}$ is stationary.

Kaplan's lead time model is the most widely adopted model to describe stochastic lead times; see [16, 17, 1, 9].

**Deterministic Lead Time Model.** When all $\rho_t = L$ for a constant $L$, we know $L_t = L$ and Kaplan's model reduces to the deterministic lead time model.

When Assumption 1 is violated, the asymptotic optimality of order-up-to-$S$ policies may not hold or its proof requires new techniques. We discuss some cases and show the limitations of existing methods. As we will see, these extensions are quite complicated and we leave it for future directions; see Section 7.

## 5. Main Results

In this section, we present our main theorem and then provide a sketched proof. Note in this paper, we consider the system with lost-sales and we can also define the corresponding backlog system as an auxiliary system, where unsatisfied demand is backlogged and all other features are the same. We have the following main theorem.

**Theorem 1.** *Suppose that Assumptions 1 and 2 hold and there is an order-up-to-S policy that is optimal for the corresponding backlog inventory systems, then we have*

$$\lim_{b\to+\infty} \frac{\inf_{S\geq 0} C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L},*}(h,b)} = 1.$$

**Remark 1.** *The above theorem requires there is an order-up-to-S policy that is optimal for the backlog inventory systems. The requirement does not assume specific lead time models. Therefore, if one can prove the optimality of order-up-to-S policies in the backlog system, the corresponding asymptotic optimality in the lost-sales system holds. It has many possible applications.*

*For example, [10] and [17] proved that order-up-to-S policies are optimal for backorder inventory systems under Kaplan's lead time model for discount criterion and average criterion (after slight modification). Based on their conclusions, we could obtain the following corollary directly, which also covers deterministic lead times.*

**Theorem 2.** *Suppose the lead time process satisfies Kaplan's lead time model and Assumptions 2 holds, we have*

$$\lim_{b\to+\infty} \frac{\inf_{S\geq 0} C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L},*}(h,b)} = 1.$$

We give a sketched proof of Theorem 1 as follows and detailed proofs for lemmas are deferred to Section 6.

*Proof.* When demand is $\mathcal{D}$, per-unit holding cost is fixed $h$, and per-unit lost-sales cost is $b$, we denote the optimal newsvendor cost as

$$\mathrm{NV}(h,b) := h\mathbb{E}[(S_b - \mathcal{D})^+] + b\mathbb{E}[(\mathcal{D} - S_b)^+],$$

where the optimal solution is given by

$$S_b := \inf\left\{x : \mathbb{P}[\mathcal{D} \leq y] \geq \frac{b}{b+h}\right\}.$$

The high-level idea of the proof is to bound the costs of the lost-sales system by newsvendor costs with different per-unit lost-sales costs. Specifically, we decompose the proof into the following steps.

1. In Step I, by Lemma 4 we show

$$C^{\mathcal{L},*}(h,b) \geq \text{NV}\left(h, b/(\overline{L}+1)\right).$$

2. In Step II, by Lemma 5 we prove

$$C^{\mathcal{L},S_{b+h\overline{L}}}(h,b) \leq \text{NV}\left(h, b+h\overline{L}\right).$$

3. In Step III, by Lemma 6 we demonstrate

$$\lim_{b\to\infty} \frac{\text{NV}\left(h, b+h\overline{L}\right)}{\text{NV}\left(h, b/(\overline{L}+1)\right)} = 1.$$

By the above three steps, we have

$$1 \leq \lim_{b\to\infty} \frac{\inf_{S\geq 0} C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L},*}(h,b)} \leq \lim_{b\to\infty} \frac{\text{NV}\left(h, b+h\overline{L}\right)}{\text{NV}\left(h, b/(\overline{L}+1)\right)} = 1.$$

Thus, we complete the proof of Theorem 1. □

## 6. Proof of Theorem 1

In this section, we present the detailed proof of Theorem 1 in three steps as introduced in Section 5.

### 6.1. Step I: Bound the Optimal Cost from Below

In the following lemma, we show that the optimal cost of the lost-sales system with per-unit lost-sales cost of $b$ is bounded from below by the newsvendor cost with per-unit lost-sales cost of $b/(\overline{L}+1)$.

**Lemma 4.** *Suppose that Assumption 1 holds and there is an order-up-to-S policy that is optimal for the corresponding backlog inventory systems. We have*

$$C^{\mathcal{L},*}(h,b) \geq \text{NV}\left(h, b/(\overline{L}+1)\right).$$

The proof idea of this lemma is from [15] studying problems with constant lead times. We generalize its idea to the stochastic lead times setting directly and present the rigorous proof as follows mainly for completeness.

**Remark 2.** *When lead times are deterministic, [5] and [6] proposed a sample-path approach to establish a similar lower bound. The method is simple and powerful in many problems, but it requires the independence between $\mathcal{D}_{\lambda_t}$ and after-delivery on-hand inventory in period $\lambda_t$ to apply Jensen's inequality, which seems hard to satisfy due to the stochastic lead times. Fortunately, the imitation-based method in this proof still works under stochastic lead times.*

*Proof.* The main idea of the proof is to construct a policy $\pi^{\mathcal{B}}$ for the backorder system to imitate the behavior of the lost-sales system under its optimal policy. Without loss of generality, we assume there exists an optimal policy for the lost-sales system (Otherwise, an analogous argument works for near-optimal policy $\hat{\pi}$ such that $C^{\mathcal{L},\hat{\pi}}(h,b) \leq C^{\mathcal{L},*}(h,b) + \epsilon$ for any $\epsilon > 0$).

Let $\pi^{\mathcal{L},*}$ be the optimal policy. Under the policy $\pi^{\mathcal{L},*}$, we denote the order quantity, lost-sales quantity, and after-delivery on-hand inventory in period $t$ by $q_t^{\mathcal{L}}$, $l_t^{\mathcal{L}}$, and $I_t^{\mathcal{L}}$, respectively. Similarly, we define $q_t^{\mathcal{B}}$, $I_t^{\mathcal{B}}$ for the backorder system under the policy $\pi^{\mathcal{B}}$ to be constructed.

Now, we define the policy $\pi^{\mathcal{B}}$ as follows. In period $t$, the order quantity given by $\pi^{\mathcal{B}}$ is

$$q_t^{\mathcal{B}} = l_{t-1}^{\mathcal{L}} + q_t^{\mathcal{L}}.$$

We also require that the policy satisfies the backlogged demand in period $t$ using the order placed in period $t+1$. Note that under such a service mechanism, there may be on-hand inventory and backlogged demand simultaneously. Let $C^{\mathcal{B},*}(h,b)$ be the optimal cost of the backlog system, and we know

$$C^{\mathcal{B},\pi^{\mathcal{B}}}(h,b) \geq C^{\mathcal{B},*}(h,b). \tag{2}$$

Next, we compare the cost $C^{\mathcal{B},\pi^{\mathcal{B}}}(h,b)$ with the optimal cost of the lost-sales system.

First, we consider the period $t = 1$. Both systems start from the same initial state. The unsatisfied demand $l_1^{\mathcal{L}}$ of the lost-sales system is also the backlogged demand in the backorder system. Then, in period $t = 2$, the backorder system orders $q_2^{\mathcal{B},\pi^{\mathcal{B}}} = l_1^{\mathcal{L}} + q_2^{\mathcal{L}}$. The quantity $q_2^{\mathcal{L}}$ is just the order quantity of the lost-sales system and the additional order quantity $l_1^{\mathcal{L}}$ will be used to satisfy the demand backlogged in period 1. By a simple induction, we can see that under the policy $\pi^{\mathcal{B}}$,

1. The on-hand inventories $I_t^{\mathcal{B}}$ and $I_t^{\mathcal{L}}$ of the two systems are equal.
2. The backlogged demand $(D_t - I_t^{\mathcal{B}})^+$ in period $t$ will cause backorder costs for $L_t + 1$ periods.

Therefore, we know the two systems incur the same holding cost. Moreover, the backlogged demand will be satisfied after at most $\overline{L}+1$ periods by Assumption 1. We have

$$\begin{aligned}
&C^{\mathcal{B},\pi^{\mathcal{B}}}(h,b) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[h(I_t^{\mathcal{B}} - D_t)^+ + b\cdot(L_t+1)\cdot(D_t - I_t^{\mathcal{B}})^+\right] \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[h(I_t^{\mathcal{B}} - D_t)^+ + b\cdot(\overline{L}+1)\cdot(D_t - I_t^{\mathcal{B}})^+\right] \\
&= \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[h(I_t^{\mathcal{L}} - D_t)^+ + b\cdot(\overline{L}+1)\cdot(D_t - I_t^{\mathcal{L}})^+\right] \\
&= C^{\mathcal{L},*}(h, b(\overline{L}+1)),
\end{aligned}$$

Combine the above inequality with Eq. (2), we know

$$C^{\mathcal{L},*}(h, b(\overline{L}+1)) \geq C^{\mathcal{B},*}(h,b). \tag{3}$$

Since the above inequality holds for any non-negative $h$ and $b$, replacing $b$ with $b/(\overline{L}+1)$ gives that

$$C^{\mathcal{L},*}(h,b) \geq C^{\mathcal{B},*}(h, b/(\overline{L}+1)).$$

Next, we show $C^{\mathcal{B},*}(h,b) \geq \mathrm{NV}(h,b)$ for any $b$.

Consider the backorder system with backlog costs of $b$ per unit. By the assumption that base-stock policies are optimal in the backlogging system, the optimal cost can be attained by some order-up-to-$S$ policy.

Recall $\lambda_t \triangleq \max\{i : i + L_i \leq t\}$. $\lambda_t$ denotes the index of the period that has the latest shipment in or before period $t$. Hence, the inventory position $S$ ( under the order-up-to-$S$ policy) will all arrive in or before $t$, and inventory level $I_t$ equals $S$ minus all demand during the period from $\lambda_t$ to $t$. Therefore, we know

$$I_t = S - \sum_{i=\lambda_t}^{t-1} D_i.$$

Therefore, for $t \geq \bar{L} + 1$, we know the cost in period $t$ is

$$\begin{aligned}
&\mathbb{E}[h(I_t - D_t)^+] + \mathbb{E}[b(D_t - I_t)^+] \\
&= \mathbb{E}[h(S - \sum_{i=\lambda_t}^{t-1} D_i - D_t)^+] + \mathbb{E}[b(D_t - S + \sum_{i=\lambda_t}^{t-1} D_i)^+] \\
&= \mathbb{E}[h(S - \mathcal{D}_t)^+] + \mathbb{E}[b(\mathcal{D}_t - S)^+] \geq \mathrm{NV}(h,b).
\end{aligned}$$

By the definition of long-run average cost, we know that

$$C^{\mathcal{B},*}(h,b) \geq \mathrm{NV}(h,b).$$

For the above inequality, replacing $b$ with $b/(\overline{L}+1)$ and combining it with Eq. (3), we complete the proof. □

### 6.2. Step II: Bound the Cost of Base Stock Policies from Above

In the following lemma, we prove that under the order-up-to-$S$ policy, the cost of the lost-sales system with per-unit lost-sales cost of $b$ is bounded from above by a newsvendor cost with a per-unit lost-sales cost of $b + h\overline{L}$.

**Lemma 5.** *Under Assumption 1, we have*

$$C^{\mathcal{L},S_{b+h\overline{L}}}(h,b) \leq \mathrm{NV}\left(h, b + h\overline{L}\right).$$

*Proof.* Consider any order-up-to level $S$. Under the policy, let $I_t$, and $l_t$ be defined similarly as Lemma 4. We also denote $q_t$ the order in period $t$. Note that all quantities in this proof are defined under the order-up-to-$S$ policy and we omit the superscribe of $S$ for simplicity.

By the definition of order-up-to-$S$ policy, we know $q_i = D_{i-1} - l_{i-1}$. Moreover, the on-hand inventory $I_t$ plus the not delivered quantity $\sum_{i=\lambda_t+1}^{t} q_i$ equals $S$. It follows that

$$I_t = S - \sum_{i=\lambda_t+1}^{t} q_i = S - \sum_{i=\lambda_t+1}^{t} D_{i-1} + \sum_{i=\lambda_t+1}^{t} l_{i-1}. \tag{4}$$

Therefore, by the above equation, we know

$$\begin{aligned}
h\mathbb{E}[(I_t - D_t)^+] &= h\mathbb{E}[(S - \sum_{i=\lambda_t+1}^{t} D_{i-1} + \sum_{i=\lambda_t+1}^{t} l_{i-1} - D_t)^+] \\
&\leq h\mathbb{E}[(S - \sum_{i=\lambda_t+1}^{t} D_{i-1} - D_t)^+] + h\mathbb{E}[\sum_{i=\lambda_t+1}^{t} l_{i-1}] \\
&\leq h\mathbb{E}[(S - \mathcal{D}_t)^+] + h\sum_{i=t-\overline{L}+1}^{t} \mathbb{E}[l_{i-1}],
\end{aligned} \tag{5}$$

where the first inequality is by $(a+b)^+ \leq a^+ + b$, provided $b \geq 0$ and the last inequality is by the definition of $\mathcal{D}_t$.

On the other hand, by Eq. (4), we know

$$I_t = S - \sum_{i=\lambda_t+1}^{t} D_{i-1} + \sum_{i=\lambda_t+1}^{t} l_{i-1} \geq S - \sum_{i=\lambda_t+1}^{t} D_{i-1}.$$

It follows that

$$\mathbb{E}[l_t] = \mathbb{E}[(D_t - I_t)^+] \leq \mathbb{E}[(D_t - S + \sum_{i=\lambda_t+1}^{t} D_{i-1})^+] \leq \mathbb{E}[(\mathcal{D}_t - S)^+]. \tag{6}$$

Therefore, by Eq. (5) and Eq. (6), we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} [h\mathbb{E}[(I_t - D_t)^+] + b\mathbb{E}[(D_t - I_t)^+]] \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\left[h\mathbb{E}\left[(S - \mathcal{D}_t)^+\right] + h\sum_{i=t-\overline{L}+1}^{t}\mathbb{E}[l_{i-1}] + b\mathbb{E}[l_t]\right] \\
&\leq h\mathbb{E}[(S - \mathcal{D})^+] + (b + h\overline{L})\mathbb{E}[(\mathcal{D} - S)^+],
\end{aligned}$$

where in the last inequality we use that $\{\mathcal{D}_t, t \geq 1\}$ have the common distribution $\mathcal{D}$. Thus we proved that for any $S \geq 0$

$$C^{\mathcal{L},S}(h,b) \leq h\mathbb{E}[(S - \mathcal{D})^+] + (b + h\overline{L})\mathbb{E}[(\mathcal{D} - S)^+].$$

Taking $S = S_{b+h\overline{L}}$, we complete the proof. □

### 6.3. Step III: Establish the Robustness of the Optimal Newsvendor Cost

The following lemma shows that the optimal newsvendor cost is robust with respect to the lost-sales cost.

**Lemma 6.** *Suppose that Assumptions 1 and 2 hold, then we have*

$$\lim_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\overline{L}\right)}{\mathrm{NV}\left(h, b/(\overline{L}+1)\right)} = 1.$$

*Proof.* First, we consider the case that $\mathcal{D}$ is bounded with upper bound $\bar{D} := \sup\{x : F_{\mathcal{D}}(x) < 1\}$, where $F_{\mathcal{D}}(x)$ is the cumulative distribution function of $\mathcal{D}$. As $b$ approaches infinity, the newsvendor solution will tend to $\bar{D}$ and the optimal newsvendor cost $\mathbb{E}[h(S_b - \mathcal{D})^+ + b(\mathcal{D} - S_b)^+]$ will approach $h(\bar{D} - \mathbb{E}[\mathcal{D}])$. Therefore, we know

$$\lim_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\bar{L}\right)}{\mathrm{NV}\left(h, b/(\bar{L}+1)\right)} = \frac{\lim_{b\to\infty} \mathrm{NV}\left(h, b + h\bar{L}\right)}{\lim_{b\to\infty} \mathrm{NV}\left(h, b/(\bar{L}+1)\right)} = \frac{h(\bar{D} - \mathbb{E}[\mathcal{D}])}{h(\bar{D} - \mathbb{E}[\mathcal{D}])} = 1.$$

Next, we assume that $\mathcal{D}$ is unbounded. By Assumption 2, we know $m_{\mathcal{D}}(x) = o(x)$.

Theorem 1 in [6] states that if $\mathcal{D}$ is unbounded and $m_{\mathcal{D}}(x) = o(x)$, then we have

$$\lim_{b\to\infty} \frac{\mathrm{NV}(h,b)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} = h. \tag{7}$$

Therefore, define $v_b = (b + h\bar{L})/b \geq 1$ and by the definition of $S_{b+h\bar{L}}$, we have

$$\begin{aligned}
&h\mathbb{E}[(S_{b+h\bar{L}} - \mathcal{D})^+] + (b + h\bar{L})\mathbb{E}[(\mathcal{D} - S_{b+h\bar{L}})^+] \\
&\leq h\mathbb{E}[(S_b - \mathcal{D})^+] + (b + h\bar{L})\mathbb{E}[(\mathcal{D} - S_b)^+] \\
&= v_b\left\{v_b^{-1} h\mathbb{E}[(S_b - \mathcal{D})^+] + b\mathbb{E}[(\mathcal{D} - S_b)^+]\right\} \\
&\leq v_b \mathrm{NV}(h,b),
\end{aligned}$$

and

$$\limsup_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\bar{L}\right)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} \leq \lim_{b\to\infty} v_b \cdot \lim_{b\to\infty} \frac{\mathrm{NV}(h,b)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} = h,$$

where the equality is by Eq. (7).

On the other hand, we know

$$\begin{aligned}
\lim_{b\to\infty} \frac{\mathrm{NV}\left(h, b/(\bar{L}+1)\right)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} &= \lim_{b\to\infty} \frac{1}{\bar{L}+1} \cdot \frac{\mathrm{NV}\left(h(\bar{L}+1), b\right)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} \\
&= \frac{1}{\bar{L}+1} \cdot \lim_{b\to\infty} \frac{\mathrm{NV}\left(h(\bar{L}+1), b\right)}{F_{\mathcal{D}}^{-1}(\frac{b}{b+1})} = h.
\end{aligned}$$

Combining the above two conclusions, we obtain

$$\limsup_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\bar{L}\right)}{\mathrm{NV}\left(h, b/(\bar{L}+1)\right)} \leq \limsup_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\bar{L}\right)}{hF_{\mathcal{D}}^{-1}(\frac{b}{b+1})} \leq 1.$$

Meanwhile, we know

$$\liminf_{b\to\infty} \frac{\mathrm{NV}\left(h, b + h\bar{L}\right)}{\mathrm{NV}\left(h, b/(\bar{L}+1)\right)} \geq 1,$$

since $b + h\bar{L} \geq b/(\bar{L}+1)$. By the above two inequality, we complete the proof. □

## 7. Concluding Remarks

The inventory control of lost-sales inventory systems with stochastic lead times is notoriously hard due to the well-known curse of dimensionality. Due to their simplicity and good performance, base-stock policies are widely adopted in practice. In this paper, we prove the asymptotic optimality of base-stock policies in the lost-sales inventory systems, generalizing the existing result to the setting of stochastic lead times. The results of this paper justify the wide application of base-stock policies from a theoretical perspective and shed light on future research directions in extending the existing lost-sales inventory management learning algorithms [13, 21] to address more complex scenarios involving stochastic lead times.

For future directions, it would also be interesting to consider the problem with crossing stochastic lead times or Markov-modulated lead times, which seems to call for new techniques.

**Lead time models with crossover.** Note the asymptotic optimality of base-stock policies is due to the similar behavior of lost-sales and backlog systems in the regime of large lost-sales costs (both systems tend to avoid stock-out). Therefore, the optimal policy for the backlog system is also asymptotically optimal for the lost-sales system. All the existing methods are based on the above intuition.

However, the base-stock policies may not be optimal for the backlog systems with lead time crossover, e.g., *i.i.d.* lead times; see [20]. Therefore, we first need to study whether base-stock policies are asymptotically optimal for backlog systems, then the relation between the lost-sales and backlog systems.

**Lead time models under Markov environment.** In this paper, we prove the asymptotic optimality of constant base-stock policies under lead time models such that constant base-stock policies are optimal in backlog systems like Kaplan's model. There are also Markov-modulated lead time processes; see [17]. Under these models, it is proved that a state-dependent base-stock policy is optimal for backlog systems. Therefore, one may expect the state-dependent base-stock policy would be asymptotically optimal for the corresponding lost-sales systems.

To establish the asymptotic optimality of state-dependent base-stock policy, we need to solve many new technical challenges. For example, one may need to generalize the existing comparison results to the Markov-modulated environment, relate the costs to a closed form (may not have a newsvendor cost form), and study the limiting behavior of the closed form to obtain the final conclusion.

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