CUMCM-2024 Problem D

The Hitting Probability for Anti-Submarine Aerial Depth Charge

The application of depth charges was an important means for anti-submarine warfare during World War II. With the development of modern military technology, although torpedoes have become the primary weapon for this purpose, some countries are still developing the anti-submarine technology of depth charges due to its low cost and strong anti-interference capabilities, especially in the areas with complex seafloor topography such as straits or shallow sea.

Before an anti-submarine aircraft attacks underwater targets, a reconnaissance aircraft is usually sent out to detect the approximate location of the submarine, and then an anti-submarine aircraft, equipped with depth charges, is dispatched to execute the mission. When a submarine detects that it is being tracked by an electronic reconnaissance aircraft, it usually shuts the electronic devices and engines to enter a silent mode to conceal on the spot.

This problem uses the target coordinate system: the projection of the positioning value of the submarine's center on the sea level is set as the origin O, the due east direction is represented by the positive X-axis, the due south direction by the positive Y-axis, and the downward direction perpendicular to the seafloor by the positive Z-axis. The azimuth angle β, measured clockwise rotation from due north to the submarine heading, which can be obtained in advance under certain conditions (see Figure 1).

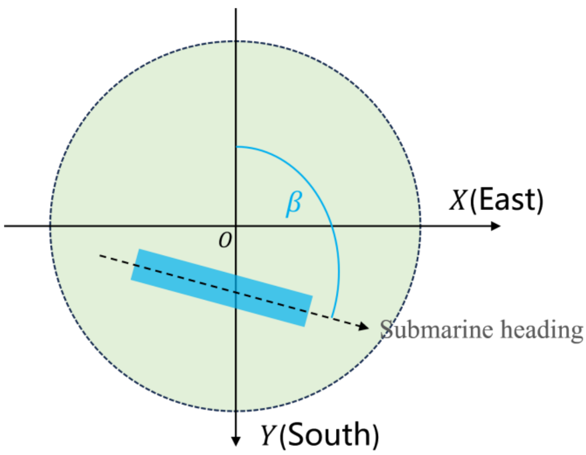

Figure 1 Schematic diagram of horizontal target positioning error and submarine heading

Due to positioning errors, the actual coordinates of the submarine's center are three independent random variables, where X and Y follow the normal distribution $N(0, \sigma^2)$ and Z follows a single-sided truncated normal distribution $N(h_0, \sigma_z^2, l)$, with a density function of

$$f_{h_0,\sigma_z,l}(v) = \frac{1}{\sigma_z} \cdot \frac{\phi\left(\dfrac{v - h_0}{\sigma_z}\right)}{1 - \Phi\left(\dfrac{l - h_0}{\sigma_z}\right)} \qquad (l < v < +\infty),$$

where h_0 is the positioning value of the depth at the center position of the submarine, l is the minimum value of the actual depth at the center position of the submarine, and ϕ and Φ are the density function and distribution function of the standard normal distribution, respectively.

For simplicity, assuming that the main body of the submarine is a cuboid, and the depth charge descends vertically once released into the water. Assuming that the depth charge is detonated using dual fuzes (trigger fuze and depth setting fuze). The depth of detonation of the depth setting fuze is pre-set. The maximum damage distance of a depth charge in seawater is called the damage radius.

A depth charge is considered to ***hit*** the submarine if one of the following conditions is met:

(1) The landing point of the depth charge is within the scale of the target plane, and the detonation depth is below the upper surface of the submarine, under such a condition, the depth charge is detonated by the trigger fuze;

(2) The landing point of the depth charge is within the scale of the target plane, and the detonation depth is above the upper surface of the submarine, but the submarine is within the damage range of the depth charge, under such a condition, the depth charge is detonated by the depth-setting fuze;

(3) The landing point of the depth charge is outside the scale of the target plane, but the submarine is within the damage range of the depth charge when it reaches the detonation depth, under such a condition, the depth charge is still detonated by the depth-setting fuze.

Please establish mathematical models to solve the following problems:

Problem 1 Consider dropping a single depth charge, assuming that there is no error in the depth positioning of the submarine's center position, and both horizontal coordinates follow a normal distribution. Please analyze the relationship between the maximal hitting probability and the dropping locations and the depth of detonation. Moreover, provide a dropping strategy that maximizes the hitting probability, along with the corresponding expression.

Calculate the maximal hitting probability under the following parameter values: The length, width, and height of the submarine are 100m, 20m, 25m, respectively. The azimuth of the submarine is 90°. The damage radius is 20m. The standard deviation of the horizontal positioning of submarine's center is $\sigma = 120$m. The depth positioning value of the submarine's center is 150 m.

Problem 2 Still consider dropping a single bomb, but there are errors in all directions of the submarine's center. Please give the expression of the hitting probability.

Please give the depth of detonation to maximize the hitting probability under the following parameter values: The depth positioning value of the submarine's center is 150m, and the standard deviation is $\sigma_z = 40$m. The minimum actual depth at the submarine's center is 120m. The other parameters are the same as those in Problem 1.

Problem 3 Due to the low hitting probability of a single depth charge, it is necessary to drop multiple depth charges to enhance the anti-submarine effect. Suppose an anti-submarine aircraft can carry nine depth charges, which the landing points form array shape on the plane with identical depth (see Figure 2). Under the parameters of Problem 2, please design the optimal bombs dropping scheme that maximizes the hitting probability (referring to at least one depth charge hitting the submarine), including the depth of detonation and the plane spacing between the landing points.

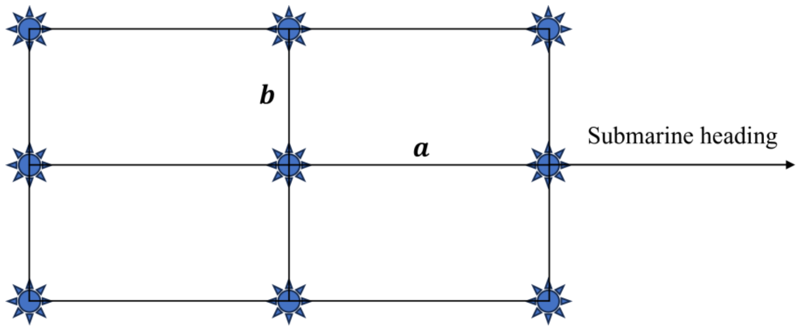

Figure 2　Schematic diagram of the plane distribution of multiple landing points